CHAINS OF THEORIES AND COMPANIONABILITY

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Abstract. The theory of fields with a countably infinite family of commuting derivations is not companionable; but if the axiom is added whereby the characteristic of the fields is zero, then the resulting theory is companionable. Each of these two theories is the union of a chain of companionable theories. In a signature with predicates, in all finite numbers of arguments, for linear dependence of vectors, the two-sorted theory of vector-spaces with their scalar-fields is companionable, and it is the union of a chain of companionable theories, but its model-companion is not the union of the model-companions of the theories in the chain. Finally, the union of a chain of non-companionable theories may be companionable.

A **theory** in a given signature is a set of sentences, in the first-order logic of that signature, that is closed under logical implication. We shall consider chains $(T_m \colon m \in \omega)$ of theories: this means

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \tag{$*$}$$

The signature of T_m will be $\mathscr{S}_m$, so automatically $\mathscr{S}_0 \subseteq \mathscr{S}_1 \subseteq \mathscr{S}_2 \subseteq \cdots$

In one motivating example, $\mathscr{S}_m$ is $\{0, 1, -, +, \cdot, \partial_0, \dots, \partial_{m-1}\}$, the signature of fields with m additional singulary function-symbols; and T_m is m-DF, the theory of fields (of any characteristic) with m commuting derivations. In this example, each T_{m+1} is a **conservative extension** of T_m, that is, $T_{m+1} \supseteq T_m$ and every sentence in T_{m+1} of signature $\mathscr{S}_m$ is already in T_m. We establish this by showing that every model of T_m expands to a model of T_{m+1}. (This condition is sufficient, but not necessary [3, §2.6, exer. 8, p. 66].) If $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, then $(K, \partial_0, \dots, \partial_m) \models (m+1)$-DF, where ∂_m is the 0-derivation.

The union of the theories m-DF can be denoted by ω-DF: it is the theory of fields with ω-many commuting derivations. Each of the theories m-DF has a *model-companion,* called m-DCF [11]; but we shall show (as Theorem 3 below) that ω-DF has no model-companion. Let us recall that a **model-companion** of a theory T is a theory T^* in the same signature such that (1) $T_\forall = {T^*}_\forall$, that is, every model of one of the theories embeds in a model of the other, and (2) T^* is **model-complete,** that is, $T^* \cup \operatorname{diag}(\mathfrak{M})$ axiomatizes a complete theory for all models $\mathfrak{M}$ of T^*. Here $\operatorname{diag}(\mathfrak{M})$ is the quantifier-free theory of $\mathfrak{M}$ with parameters: equivalently, $\operatorname{diag}(\mathfrak{M})$ is the theory of all structures in which $\mathfrak{M}_M$ embeds. (These notions, with historical references, are reviewed further in [11].) A theory has at most one model-companion, by an argument with interwoven elementary chains.

Let m-DF_0 be m-DF with the additional requirement that the field have characteristic 0. Then m-DF_0 has a model-companion, called m-DCF_0 [6]. We shall show (as Theorem 6 below) that m-$\mathrm{DCF}_0 \subseteq (m+1)$-$\mathrm{DCF}_0$. It will follow then that the union ω-DF_0 of the m-DF_0 has a model-companion, which is the union of the m-DCF_0. This is by the following general result, which has been observed also by Alice Medvedev [7, 8]. Again, the theories T_k are as in $(*)$ above.

Theorem 1. *Suppose each theory T_k has a model-companion ${T_k}^*$, and*

$${T_0}^* \subseteq {T_1}^* \subseteq {T_2}^* \subseteq \cdots \tag{$\dagger$}$$

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Then the theory $\bigcup_{k\in\omega} T_k$ *has a model-companion, namely* $\bigcup_{k\in\omega} {T_k}^*$.

Proof. Write U for $\bigcup_{k\in\omega} T_k$, and U^* for $\bigcup_{k\in\omega} {T_k}^*$. Suppose $\mathfrak{A} \models U$, and Γ is a finite subset of $U^* \cup \operatorname{diag}(\mathfrak{A})$. Then Γ is a subset of ${T_k}^* \cup \operatorname{diag}(\mathfrak{A} \upharpoonright \mathscr{S}_k)$ for some k in ω, and also $\mathfrak{A} \upharpoonright \mathscr{S}_k \models T_k$. Since $({T_k}^*)_\forall \subseteq T_k$, we conclude that Γ is consistent. Therefore $U^* \cup \operatorname{diag}(\mathfrak{A})$ is consistent. Thus ${U^*}_\forall \subseteq U$. By symmetry $U_\forall \subseteq U^*$.

Similarly, if $\mathfrak{B} \models U^*$, then ${T_k}^* \cup \operatorname{diag}(\mathfrak{B} \upharpoonright \mathscr{S}_k)$ axiomatizes a complete theory in each case, and therefore $U^* \cup \operatorname{diag}(\mathfrak{B})$ is complete. □

The foregoing proof does not require that the signatures $\mathscr{S}_k$ form a chain, but needs only that every finite subset of $\bigcup_{k\in\omega} \mathscr{S}_k$ be included in some $\mathscr{S}_k$. This is the setting for Medvedev's [8, Prop. 2.4, p. 6], which then has the same proof as the foregoing. Also in Medvedev's setting, each ${T_{k+1}}^*$ is a conservative extension of ${T_k}^*$; but only the weaker assumption ${T_k}^* \subseteq {T_{k+1}}^*$ is needed in the proof.

Medvedev notes that many properties that the theories T_k might have are 'local' and are therefore preserved in $\bigcup_{k\in\omega} T_k$: examples are completeness, elimination of quantifiers, stability, and simplicity. In her main application, $\mathscr{S}_n$ is the signature of fields with singulary operation-symbols $\sigma_{m/n!}$, where $m \in \mathbb{Z}$; and T_n is the theory of fields on which the $\sigma_{m/n!}$ are automorphisms such that

$$\sigma_{k/n!} \circ \sigma_{m/n!} = \sigma_{(k+m)/n!}.$$

Then T_n includes the theory S_n of fields with the single automorphism $\sigma_{1/n!}$. Using [12, §1] (which is based on [3, ch. 5]), we may observe at this point that reduction of models of T_n to models of S_n is actually an equivalence of the categories $\operatorname{Mod}^{\subseteq}(T_n)$ and $\operatorname{Mod}^{\subseteq}(S_n)$, whose objects are models of the indicated theories, and whose morphisms are embeddings. We thus have at hand a (rather simple) instance of the hypothesis of the following theorem.

Theorem 2. *Suppose* (I, J) *is a bi-interpretation of theories* S *and* T *such that* I *is an equivalence of the categories* $\operatorname{Mod}^{\subseteq}(S)$ *and* $\operatorname{Mod}^{\subseteq}(T)$. *If* S *has the model-companion* S^*, *and* $S \subseteq S^*$, *then* T *also has a model-companion, which is the theory of those models* $\mathfrak{B}$ *of* T *such that* $J(\mathfrak{B}) \models S^*$.

Proof. The class of models $\mathfrak{B}$ of T such that $J(\mathfrak{B}) \models S^*$ is elementary. Let T^* be its theory. Then $T \subseteq T^*$. Suppose $\mathfrak{B} \models T$. Then $J(\mathfrak{B}) \models S$, so $J(\mathfrak{B})$ embeds in a model $\mathfrak{A}$ of S^*. Consequently $I(J(\mathfrak{B}))$ embeds in $I(\mathfrak{A})$. Also $I(\mathfrak{A}) \models T^*$, since $\mathfrak{A} \cong J(I(\mathfrak{A}))$. Since also $\mathfrak{B} \cong I(J(\mathfrak{B}))$, we conclude that $\mathfrak{B}$ embeds in a model of T^*. Finally, T^* is model-complete. Indeed, suppose now $\mathfrak{B}$ and $\mathfrak{C}$ are models of T^* such that $\mathfrak{B} \subseteq \mathfrak{C}$. Then $J(\mathfrak{B})$ embeds in $J(\mathfrak{C})$, and these structures are models of S^*, so the embedding is elementary. Therefore an elementary embedding of $I(J(\mathfrak{B}))$ in $I(J(\mathfrak{C}))$ is induced. By the equivalence of the categories, $\mathfrak{B} \preccurlyeq \mathfrak{C}$. □

In the present situation, the theory S_n has a model-companion [5, 1]; let us denote this by ACFA_n. By the theorem then, T_n has a model-companion ${T_n}^*$, which is axiomatized by $T_n \cup \mathrm{ACFA}_n$. We have $\mathrm{ACFA}_n \subseteq {T_{n+1}}^*$ by [1, 1.12, Cor. 1, p. 3013]. By Theorem 1 then, $\bigcup_{n\in\omega} T_n$ has a model-companion, which is the union of the ${T_n}^*$. Medvedev calls this union $\mathbb{Q}\mathrm{ACFA}$; she shows for example that it preserves the simplicity of the ACFA_n, as noted above, though it does not preserve their supersimplicity.

The following is similar to the result that the theory of fields with a derivation *and* an automorphism (of the field-structure only) has no model-companion [10]. The obstruction lies in positive characteristics p, where all derivatives of elements with p-th roots must be 0.

Theorem 3. *The theory* ω-DF *has no model-companion.*

Proof. We use that an $\forall\exists$ theory T has a model-companion if and only if the class of its *existentially closed* models is elementary, and in this case the model-companion is the theory of this class [2]. (A model $\mathfrak{A}$ of T is an **existentially closed** model, provided that if $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$, that is, all quantifier-free formulas over A that are soluble in $\mathfrak{B}$ are soluble in $\mathfrak{A}$.) For each n in ω, the theory ω-DF has an existentially closed model $\mathfrak{A}_n$, whose underlying field includes $\mathbb{F}_p(\alpha)$, where α is transcendental; and in this model,

$$\partial_k\alpha = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then α has no p-th root in $\mathfrak{A}_n$. Therefore, in a non-principal ultraproduct of the $\mathfrak{A}_n$, α has no p-th root, although $\partial_n\alpha = 0$ for all n in ω, so that α does have a p-th root in some extension. Thus the ultraproduct is not an existentially closed model of ω-DF. Therefore the class of existentially closed models of ω-DF is not elementary. $\square$

It follows then by Theorem 1 that m-DCF $\nsubseteq$ $(m+1)$-DCF for at least one m. We could contrive examples to show this independently; but this by itself would not be enough to establish the last theorem. For, by the results of [12], it is possible for each T_k to have a model-companion ${T_k}^*$, while $\bigcup_{k\in\omega} T_k$ has a model-companion that is not $\bigcup_{k\in\omega} {T_k}^*$. We may even require T_{k+1} to be a conservative extension of T_k.

Indeed, if $k > 0$, then in the notation of [12], VS_k is the theory of vector-spaces with their scalar-fields in the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *, P^k\}$, where $\circ$ is multiplication of scalars, and $*$ is the action of the scalar-field on the vector-space, and P^k is k-ary linear dependence. In particular, P^2 may written also as $\|$. Then VS_k has a model-companion, VS_k^*, which is the theory of k-dimensional vector-spaces over algebraically closed fields [12, Thm 2.3]. Let $\mathrm{VS}_\omega = \bigcup_{1\leqslant k<\omega} \mathrm{VS}_k$. (This was called VS_∞ in [12].) This theory has the model-companion ${\mathrm{VS}_\omega}^*$, which is the theory of infinite-dimensional vector-spaces over algebraically closed fields [12, Thm 2.4].

Theorem 4. *If $1 \leqslant n < \omega$, let*

$$T_n = \bigcup_{1\leqslant k\leqslant n} \mathrm{VS}_k\,.$$

Then T_n has a model-companion ${T_n}^*$, which is axiomatized by $\mathrm{VS}_n^* \cup T_n$. Also T_{n+1} is a conservative extension of T_n. However, the model-companion ${\mathrm{VS}_\omega}^*$ of the union VS_ω of the chain $(T_n\colon 1 \leqslant n < \omega)$ is not the union of the ${T_n}^*$.

Proof. Every vector-space can be considered as a model of any VS_k and hence of any T_k. In particular, T_{n+1} is a conservative extension of T_n. The relation P^1 is defined by $\boldsymbol{x} = \mathbf{0}$. Let $\mathrm{VS}_n^{\mathrm{m}}$ be axiomatized by VS_n and the requirement that the space have dimension at least n. If $n > 2$, then there are existential formulas that, in each model of $\mathrm{VS}_n^{\mathrm{m}}$, define the relation $\|$ and its complement [12, §2, p. 431]. Similarly, if $2 \leqslant k < n-1$, then, using existential formulas, we can define P^{k+1} and its complement in models of $\mathrm{VS}_k \cup \mathrm{VS}_n^{\mathrm{m}}$: indeed, $\neg P^{k+1}\boldsymbol{x}_0\cdots\boldsymbol{x}_k$ is equivalent to $\exists(\boldsymbol{x}_{k+1},\dots,\boldsymbol{x}_{n-1})\ P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{n-1}$, and $P^{k+1}\boldsymbol{x}_0\cdots\boldsymbol{x}_k$ is equivalent to

$$\exists(\boldsymbol{x}_{k+1},\dots,\boldsymbol{x}_n)\ \Bigl(P^k\boldsymbol{x}_1\cdots\boldsymbol{x}_k \vee \Bigl(\neg P^n\boldsymbol{x}_1\cdots\boldsymbol{x}_n \wedge \bigwedge_{j=k}^{n} P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{j-1}\boldsymbol{x}_{j+1}\cdots\boldsymbol{x}_n\Bigr)\Bigr).$$

Therefore reduction from models of T_n to models of $\mathrm{VS}_n^{\mathrm{m}}$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(T_n)$ and $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_n^{\mathrm{m}})$ [12, Lem 1.1]. Since $\mathrm{VS}_n \subseteq \mathrm{VS}_n^{\mathrm{m}}$ and every model of VS_n embeds in a model of $\mathrm{VS}_n^{\mathrm{m}}$, the two theories have the same model-companion, namely $\mathrm{VS}_n{}^*$. Therefore $\mathrm{VS}_n{}^* \cup T_n$ axiomatizes $T_n{}^*$, by Theorem 2. The rest has already been noted. □

Let $\mathrm{VS}_n^{\mathrm{r}}$ comprise the sentences of VS_n having one-sorted signature $\{\mathbf{0}, -, +, P^n\}$. If $n > 1$, then reduction of VS_n to $\mathrm{VS}_n^{\mathrm{r}}$ is an equivalence of categories $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_n)$ and $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_n^{\mathrm{r}})$, and therefore $\mathrm{VS}_n^{\mathrm{r}}$ is companionable. (This was mentioned at [12, p. 431], but the details, and in particular Theorem 2, were not spelled out.) So there is an alternative, one-sorted version of the last theorem, where T_n is $\bigcup_{1 \leqslant k \leqslant n} \mathrm{VS}_k^{\mathrm{r}}$.

The implication $A \Rightarrow B$ in the following is used implicitly at [1, 1.12, p. 3013] to establish the result used above, that if (K, σ) is a model of ACFA, then so is (K, σ^m), assuming $m \geqslant 1$.

Theorem 5. *Assuming as usual $T_0 \subseteq T_1$, where each T_k has signature $\mathscr{S}_k$, we consider the following conditions.*

A. For every model $\mathfrak{A}$ of T_1 and model $\mathfrak{B}$ of T_0 such that

$$\mathfrak{A} \restriction \mathscr{S}_0 \subseteq \mathfrak{B}, \tag{‡}$$

there is a model $\mathfrak{C}$ of T_1 such that

$$\mathfrak{A} \subseteq \mathfrak{C}, \qquad\qquad \mathfrak{B} \subseteq \mathfrak{C} \restriction \mathscr{S}_0. \tag{§}$$

B. The reduct to $\mathscr{S}_0$ of every existentially closed model of T_1 is an existentially closed model of T_0.

C. T_0 has the Amalgamation Property: *if one model embed in two others, then those two in turn embed in a fourth model, compatibly with the original embeddings.*

D. T_1 is $\forall\exists$ (so that every model embeds in an existentially closed model).

We have the two implications

$$A \implies B, \qquad\qquad B \ \&\ C \ \&\ D \implies A,$$

but there is no implication among the four conditions that does not follow from these. This is true, even if T_1 is required to be a conservative extension of T_0.

Proof. Suppose A holds. Let $\mathfrak{A}$ be an existentially closed model of T_1, and let $\mathfrak{B}$ be an arbitrary model of T_0 such that (‡) holds. By hypothesis, there is a model $\mathfrak{C}$ of T_1 such that (§) holds. Then $\mathfrak{A} \preccurlyeq_1 \mathfrak{C}$, and therefore $\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{C} \restriction \mathscr{S}_0$, and *a fortiori* $\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{B}$. Therefore $\mathfrak{A} \restriction \mathscr{S}_0$ must be an existentially closed model of T_0. Thus B holds.

Suppose conversely B holds, along with C and D. Let $\mathfrak{A} \models T_1$ and $\mathfrak{B} \models T_0$ such that (‡) holds. We establish the consistency of $T_1 \cup \mathrm{diag}(\mathfrak{A}) \cup \mathrm{diag}(\mathfrak{B})$. It is enough to show the consistency of

$$T_1 \cup \mathrm{diag}(\mathfrak{A}) \cup \{\exists \boldsymbol{x}\ \varphi(\boldsymbol{x})\}, \tag{¶}$$

where φ is an arbitrary quantifier-free formula of $\mathscr{S}_0(A)$ that is soluble in $\mathfrak{B}$. By D, there is an existentially closed model $\mathfrak{C}$ of T_1 that extends $\mathfrak{A}$. By B then, $\mathfrak{C} \restriction \mathscr{S}_0$ is an existentially closed model of T_0 that extends $\mathfrak{A} \restriction \mathscr{S}_0$. By C, both $\mathfrak{B}$ and $\mathfrak{C} \restriction \mathscr{S}_0$ embed over $\mathfrak{A} \restriction \mathscr{S}_0$ in a model of T_0. In particular, φ will be soluble in this model. Therefore φ is already soluble in $\mathfrak{C} \restriction \mathscr{S}_0$ itself. Thus $\mathfrak{C}$ is a model of (¶). Therefore A holds.

The foregoing arguments eliminate the five possibilities marked X on the table below, where 0 means false, and 1, true. We give examples of each of the remaining cases, numbered according to the table. In each example, T_0 will be the reduct of T_1 to $\mathscr{S}_0$. We shall denote by $\mathscr{S}_{\mathrm{f}}$ the signature $\{+, \cdot, -, 0, 1\}$ of fields; and by $\mathscr{S}_{\mathrm{vs}}$, the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *\}$ of vector-spaces as two-sorted structures.

	1	X	2	3	4	X	5	6	7	X	8	9	10	X	X	11
A	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
B	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
C	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
D	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

1. We first give an example in which none of the four lettered conditions hold. Let $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}} \cup \{a, b\}$ and $\mathscr{S}_1 = \mathscr{S}_0 \cup \{c\}$. Let T_1 be the theory of fields of characteristic p with distinguished elements a, b, and c such that $\{a, c\}$ or $\{b, c\}$ is p-independent, and if $\{b, c\}$ is p-independent, then so is $\{b, c, d\}$ for some d. Then T_0 is the theory of fields of characteristic p in which, for some c, $\{a, c\}$ or $\{b, c\}$ is p-independent, and if $\{b, c\}$ is p-independent, then so is $\{b, c, d\}$ for some d. The negations of the four lettered conditions are established as follows. Throughout, a, b, c, and d will be algebraically independent over $\mathbb{F}_p$.

$\neg A$. We have

$$(\mathbb{F}_p(a, b^{1/p}, c), a, b, c) \models T_1, \qquad\qquad (\mathbb{F}_p(a, b^{1/p}, c^{1/p}), a, b) \models T_0,$$

but if $(\mathbb{F}_p(a, b^{1/p}, c), a, b, c)$ is a substructure of a model (K, a, b, c) of T_1, then K cannot contain $c^{1/p}$.

$\neg B$. T_0 has no existentially closed models, since an element of a model that is p-independent from a or b will always have a p-th root in some extension. Similarly, no model of T_1 in which $\{a, c\}$ is not p-independent is existentially closed. But T_1 does have existentially closed models, which are just the separably closed fields of characteristic p with p-basis $\{a, c\}$ and with an additional element b.

$\neg C$. T_0 does not have the Amalgamation Property, since $(\mathbb{F}_p(a, b^{1/p}, c), a, b)$ and $(\mathbb{F}_p(a^{1/p}, b, c, d), a, b)$ are models that do not embed in the same model over the common substructure $(\mathbb{F}_p(a, b, c), a, b)$, which is a model of T_0.

$\neg D$. T_1 is not $\forall\exists$, since, as we have already noted, models in which $\{a, c\}$ is not p-independent do not embed in existentially closed models.

2. For an example of the column headed by 2 in the table, we let $\mathscr{S}_0$ and $\mathscr{S}_1$ be as in 1; but now T_1 is the theory of fields of characteristic p with distinguished elements a, b, and c such that $\{a, c, d\}$ or $\{b, c, d\}$ is p-independent $\{b, c, d\}$ for some d. This ensures that T_1 has no existentially closed models, so B holds vacuously; but the other three conditions still fail.

3. T_0 and T_1 are the same theory, so A and B hold trivially; and this theory is the theory of vector-spaces of dimension at least 2, in the signature of vector-spaces, so it neither has the Amalgamation Property, nor is $\forall\exists$.

4. T_1 is DF_p with the additional requirement that the field have p-dimension at least 2; and $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}}$, so T_0 is the theory of fields of characteristic p with p-dimension at least 2. The latter theory has the Amalgamation Property; but the other conditions fail. Indeed, let $(\mathbb{F}_p(a, b), D)$ be the model of T_1 in which $Da = 1$ and $Db = 0$: then the field $\mathbb{F}_p(a, b)$ embeds in $\mathbb{F}_p(a^{1/p}, b)$, which is a model of T_0, but D does not extend to this field. Also, T_0 has no existentially closed models; but T_1 does, and indeed it has a model-companion, namely DCF_p. Also T_1 is not $\forall\exists$, since T_0 is not: there is a chain of models of the latter, whose union is not a model, and we can make the structures in the chain into models of T_1 by adding the zero derivation.

5. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}}$, and $\mathscr{S}_1 = \mathscr{S}_0 \cup \{a\}$. T_1 is the theory of fields of characteristic p with distinguished element a, which is p-independent from another element; so T_0

is (as in 4) the theory of fields of characteristic p with p-dimension at least 2. Here T_1 has no existentially closed models, so B holds trivially.

6. T_0 and T_1 are the same, namely the theory of fields of characteristic p of positive p-dimension, in the signature of fields, so this theory has the Amalgamation Property, but is not $\forall\exists$.

7. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{vs}}$, $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\|, \boldsymbol{a}, \boldsymbol{b}\}$, and T_1 is axiomatized by $\mathrm{VS}_2 \cup \{\boldsymbol{a} \nparallel \boldsymbol{b}\}$, so it is $\forall\exists$. Then T_0 is the theory of vector-spaces of dimension at least 2. As in Theorem 4 above, T_1 has a model-companion, namely the theory of vector-spaces over algebraically closed fields with basis $\{\boldsymbol{a}, \boldsymbol{b}\}$. But T_0 has no existentially closed models, since for all independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in some model, the equation $x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0}$ is always soluble in some extension. Thus B fails. Then T_0 also does not have the Amalgamation Property, since the solutions of the given equation may satisfy $2x^2 = y^2$ in one extension, but $3x^2 = y^2$ in another. Similarly, A fails, since the reduct to $\mathscr{S}_0$ of a model of T_1 may embed in a model of T_0 in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

8. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{vs}} \cup \{\|\}$, $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\boldsymbol{a}, \boldsymbol{b}\}$, and T_1 is axiomatized by VS_2 together with

$$\forall x\ \forall y\ (x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0} \to 2x^2 = y^2). \tag{$\|$}$$

Then T_0 is the theory of vector-spaces such that either the dimension is at least 2, or the scalar field contains $\sqrt{2}$. As in 7, T_0 does not have the Amalgamation Property. The theory T_1 is $\forall\exists$. It also has the model $(\mathbb{Q} * \boldsymbol{a} \oplus \mathbb{Q} * \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{b})$, and $\mathbb{Q} * \boldsymbol{a} \oplus \mathbb{Q} * \boldsymbol{b}$ embeds in the model $\mathbb{Q}(\sqrt{2}, \sqrt{3}) * \boldsymbol{a}$ of T_0 when we let $\boldsymbol{b} = \sqrt{3} * \boldsymbol{a}$; but then the latter space embeds in no space in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are as required by ($\|$). So A fails. Finally, T_1 has a model-companion, axiomatized by ${\mathrm{VS}_2}^*$ together with

$$\exists x\ \exists y\ (x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0} \land 2x^2 = y^2 \land x \neq 0);$$

and T_0 has a model-companion, which is just ${\mathrm{VS}_2}^*$; so B holds.

9. T_0 and T_1 are both VS_1.

10. $T_1 = \mathrm{DF}_p$, and T_0 is the reduct to $\mathscr{S}_{\mathrm{f}}$, namely field-theory in characteristic p.

11. T_0 and T_1 are both field-theory. □

Now let $\omega\text{-DCF}_0 = \bigcup_{m\in\omega} m\text{-DCF}_0$.

Theorem 6. *For all m in ω,*

$$m\text{-DCF}_0 \subseteq (m+1)\text{-DCF}_0.$$

Therefore ω-DF_0 has a model-companion, which is ω-DCF_0. This theory admits full elimination of quantifiers, is complete, and is properly stable.

Proof. Suppose $(L, \partial_0, \dots, \partial_{m-1})$ is a model of m-DF_0, and L has a subfield K that is closed under the ∂_i (where $i < m$), and there is also a derivation ∂_m on K such that $(K, \partial_0 \restriction K, \dots, \partial_{m-1} \restriction K, \partial_m)$ is a model of $(m+1)$-DF_0. We shall include $(L, \partial_0, \dots, \partial_{m-1})$ in another model of m-DF_0, namely a model that expands to a model of $(m+1)$-DF_0 that extends $(K, \partial_0, \dots, \partial_m)$. By the last theorem, it will follow that m-$\mathrm{DCF}_0 \subseteq (m+1)$-$\mathrm{DCF}_0$. Since m is arbitrary, it will follow by Theorem 1 that ω-DCF_0 is the model-companion of ω-DF_0.

If $K = L$, we are done. So suppose $a \in L \smallsetminus K$. We shall define an extension $K(a^\xi \colon \xi \in \omega^{m+1})$ of K, and for each i in $m+1$, we shall define a derivation $\tilde{\partial}_i$ on this extension so that

$$\tilde{\partial}_i \restriction K = \partial_i \restriction K. \tag{$**$}$$

For each σ in ω^{m+1}, we shall require

$$\sigma(m) = 0 \implies a^\sigma = {\partial_0}^{\sigma(0)} \cdots {\partial_{m-1}}^{\sigma(m-1)} a. \tag{$\dagger\dagger$}$$

If $i < m+1$, let $\boldsymbol{i}$ denote the element of ω^{m+1} that takes the value 1 at i and 0 elsewhere. Then for each σ in ω^{m+1}, we shall require

$$\sigma(i) > 0 \implies \tilde{\partial}_i a^{\sigma-\boldsymbol{i}} = a^\sigma, \tag{‡‡}$$

$$\sigma(m) > 0 \wedge a^{\sigma-\boldsymbol{m}} \notin K(a^\xi\colon \xi < \sigma - \boldsymbol{m})^{\mathrm{alg}} \implies a^\sigma \notin L(a^\xi\colon \xi < \sigma)^{\mathrm{alg}}. \tag{§§}$$

These conditions ensure that the derivations $\tilde{\partial}_i$, if they do exist, are unique.

This uniqueness of the $\tilde{\partial}_i$ would be ensured, if the field L in the conclusion of (§§) were replaced with K. But the condition (§§), as it is, along with (∗∗), (††), and (‡‡), will ensure that, if $i < m$, then $\tilde{\partial}_i$ will agree with ∂_i wherever they are both defined, that is, on $K(a^\xi\colon \xi \in \omega^{m+1}) \cap L$. Indeed, the conditions (∗∗), (††), and (‡‡) ensure this agreement on $K(a^\xi\colon \xi(m) = 0)$. Moreover, we have

$$K(a^\xi\colon \xi(m) = 0) = K(a^\xi\colon \xi \in \omega^{m+1}) \cap L.$$

For, suppose $\sigma(m) > 0$ and $a^\sigma \in L$. Then by (§§) we have $a^{\sigma-\boldsymbol{m}} \in K(a^\xi\colon \xi < \sigma - \boldsymbol{m})^{\mathrm{alg}}$, and therefore, by (∗∗) and (‡‡), applying $\tilde{\partial}_m$ to $a^{\sigma-\boldsymbol{m}}$ yields $a^\sigma \in K(a^\xi\colon \xi < \sigma)$. Repeating this result, we obtain that a^σ is a rational function over $K(a^\xi\colon \xi(m) = 0)$ of certain a^η, where $\eta < \sigma$; and these a^η are algebraically independent over L. Since $a^\sigma \in L$, we conclude $a^\sigma \in K(a^\xi\colon \xi(m) = 0)$.

The last observation means that, once we have constructed $K(a^\xi\colon \xi \in \omega^{m+1})$ as desired, then, if $L \smallsetminus K(a^\xi\colon \xi \in \omega^{m+1})$ is nonempty, we can repeat the process, using an element of this difference in place of a. Thus, ultimately, we shall obtain the desired model of $(m+1)$-DF_0 whose universe includes L.

We shall build up $K(a^\xi\colon \xi \in \omega^{m+1})$ recursively, and we shall (simultaneously) establish by induction that the desired conditions are satisfied. We shall use the ordering of the σ in ω^{m+1} determined by the left-lexicographic ordering of

$$(\sigma(m), \sigma(0) + \dots + \sigma(m-1), \sigma(0), \sigma(1), \dots, \sigma(m-2)).$$

Then ω^{m+1} has the order-type of ω itself, and we shall have

$$K(a^\xi\colon \xi \in \omega^{m+1}) = \bigcup_{\tau \in \omega^{m+1}} K(a^\xi\colon \xi < \tau).$$

When $\tau = (0, \dots, 0, 1)$, then, using (††) as a definition, we have the field $K(a^\xi\colon \xi < \tau)$ as desired. Suppose we have this field as desired for some τ in ω^{m+1} such that $\tau(m) > 0$. In particular, for all i in $m+1$, we have $\tilde{\partial}_i$ as a derivation from $K(a^\xi\colon \xi + \boldsymbol{i} < \tau)$ to $K(a^\xi\colon \xi < \tau)$, and the conditions (‡‡) and (§§) hold for all σ such that $\sigma < \tau$. For defining a^τ, there are two cases to consider:

1. If $a^{\tau-\boldsymbol{m}}$ is not algebraic over $K(a^\xi\colon \xi < \tau - \boldsymbol{m})$, then we let a^τ be transcendental over $L(a^\xi\colon \xi < \tau)$, as required by (§§). We are then free to define $\tilde{\partial}_m a^{\tau-\boldsymbol{m}}$ as a^τ.
2. If $a^{\tau-\boldsymbol{m}}$ is algebraic over $K(a^\xi\colon \xi < \tau - \boldsymbol{m})$, then $\tilde{\partial}_m a^{\tau-\boldsymbol{m}}$ is determined as an element of $K(a^\xi\colon \xi < \tau)$, and we let a^τ be this element.

We now must check that, when $i < m$ and $\tau(i) > 0$, we can define $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ as a^τ. Again we consider two cases.

1. Suppose $a^{\tau-\boldsymbol{i}}$ is algebraic over $K(a^\xi\colon \xi < \tau - \boldsymbol{i})$. Then $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ is determined as an element of $K(a^\xi\colon \xi < \tau)$. Thus the value of the bracket $[\tilde{\partial}_i, \tilde{\partial}_m]$ at $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ is determined. But also, by (§§), $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ must be algebraic over $K(a^\xi\colon \xi < \tau - \boldsymbol{i} - \boldsymbol{m})$. Since the bracket is 0 on this field, it is 0 at $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ as well [11, Lem. 4.2].
2. If $a^{\tau-\boldsymbol{i}}$ is transcendental over $K(a^\xi\colon \xi < \tau - \boldsymbol{i})$, then since we are given $\tilde{\partial}_i$ as a derivation whose domain is this field, we are free to define $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ as a^τ.

Thus we have obtained $K(a^\xi\colon \xi \leqslant \tau)$ as desired. Therefore ω-DF$_0$ has the model-companion ω-DCF$_0$.

As noted, ω-DCF$_0$ inherits quantifier-elimination, completeness, and stability from the m-DCF$_0$, which have these properties [6]. Although each m-DCF$_0$ is actually ω-stable, ω-DCF$_0$ is not even superstable, since if A is a set of constants (in the sense that all of their derivatives are 0), then as σ ranges over A^ω, the sets $\{\partial_m x = \sigma(m)\colon m \in \omega\}$ belong to distinct complete types. □

We may note that, in the foregoing proof, we cannot use Condition A of Theorem 5 in the stronger form in which the structure $\mathfrak{C}$ is required to be a mere *expansion* to $\mathscr{S}_1$ of $\mathfrak{B}$:

Theorem 7. *If $m > 0$, there is a model of m-DF$_0$ that does not expand to a model of $(m+1)$-DF$_0$.*

Proof. We generalize the example of [4] repeated in [9, Ex. 1.2, p. 927]. Suppose K is a pure transcendental extension $\mathbb{Q}(a^\sigma\colon \sigma \in \omega^{m+1})$ of $\mathbb{Q}$, and $\partial_i a^\sigma = a^{\sigma+\boldsymbol{i}}$. Let L be the pure transcendental extension $K(b^\tau\colon \tau \in \omega^{m-1})$ of K, and if $i < m-1$, let $\partial_i b^\tau = b^{\tau+\boldsymbol{i}}$, while $\partial_{m-1} b^\tau = a^{(\tau,0,0)}$. Suppose ∂_m extends to L as well. We have

$$\partial_{m-1}{\partial_m}^k b^\tau = {\partial_m}^k \partial_{m-1} b^\tau = {\partial_m}^k a^{(\tau,0,0)} = a^{(\tau,0,k)},$$

and these are all algebraically independent over $\mathbb{Q}(a^\sigma\colon \sigma \in \omega^{m+1} \land \sigma(m-1) > 0)$. However, $\partial_{m-1}x$ is algebraic over this field whenever x is algebraic over K. Thus all of the ${\partial_m}^k b^\tau$ are algebraically independent over K; in particular, when $k > 0$, they do not belong to L. □

Finally, the union of a chain of non-companionable theories may be companionable:

Theorem 8. *In the signature $\{f\} \cup \{c_k\colon k \in \omega\}$, where f is a singulary operation-symbol and the c_k are constant-symbols, let T_0 be axiomatized by the sentences*

$$\forall x\ \forall y\ (fx = fy \to x = y)$$

and, for each k in ω,

$$\forall x\ (f^{k+1}x \neq x), \quad \forall x\ (fx = c_k \to x = c_{k+1}), \quad fc_{k+2} = c_{k+1} \to fc_{k+1} = c_k.$$

For each n in ω, let T_{n+1} be axiomatized by

$$T_n \cup \{fc_{n+1} = c_n\}.$$

Then

1. *each T_n is universally axiomatized, and* a fortiori *$\forall\exists$, so it does have existentially closed models;*
2. *each T_n has the Amalgamation Property;*
3. *every existentially closed model of T_{n+1} is an existentially closed model of T_n;*
4. *no T_n is companionable;*
5. *$\bigcup_{n\in\omega} T_n$ is companionable.*

Proof. Let $\mathfrak{A}_m$ be the model of T_0 with universe $\omega \times \omega$ such that

$$f^{\mathfrak{A}_m}(k,\ell) = (k,\ell+1), \qquad {c_k}^{\mathfrak{A}_m} = \begin{cases} (k-m,0), & \text{if } k > m,\\ (0,m-k), & \text{if } k \leqslant m.\end{cases}$$

Let $\mathfrak{A}_\omega$ be the model of T_0 with universe $\mathbb{Z}$ such that

$$f^{\mathfrak{A}_\omega} k = k+1, \qquad {c_k}^{\mathfrak{A}_\omega} = -k.$$

Then $\mathfrak{A}_m$ is a model of each T_k such that $k \leqslant m$; and $\mathfrak{A}_\omega$ is a model of each T_k. Moreover, each model of T_k consists of a copy of some $\mathfrak{A}_\beta$ such that $k \leqslant \beta \leqslant \omega$, along with some (or no) disjoint copies of ω and $\mathbb{Z}$ in which f is interpreted as $x \mapsto x+1$. Conversely, every structure of this form is a model of T_k. The β such that $\mathfrak{A}_\beta$ embeds in a given model of T_k is uniquely determined by that model. Consequently T_k has the Amalgamation Property. Also, a model of T_k is an existentially closed model if and only if includes no copies of ω (outside the embedded $\mathfrak{A}_\beta$): This establishes that every existentially closed model of T_{k+1} is an existentially closed model of T_k.

The existentially closed models of T_k are those models that omit the type $\{\forall y\ fy \neq x\} \cup \{x \neq c_j \colon j \in \omega\}$. In particular, $\mathfrak{A}_m$ is an existentially closed model of T_k, if $k \leqslant m$; but $\mathfrak{A}_m$ is elementarily equivalent to a structure that realizes the given type. Thus T_k is not companionable.

Finally, the model-companion of $\bigcup_{k\in\omega} T_k$ is axiomatized by this theory, together with $\forall x\ \exists y\ fy = x$. □

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