Topology

David Pierce

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Matematics Department
Mimar Sinan Fine Arts University
Istanbul
http://mat.msgsu.edu.tr/~dpierce/

These notes are based on my lectures in a graduate course of topology given in the fall semester of 2016–7. The notes are certainly not guaranteed to be free of error. I may edit them in future, as well as add to them. There are some explicitly labelled exercises, to be solved as homework. Grammatically, an exercise can be a question, a command, or a statement; it is then to be answered, obeyed, or proved, respectively. Elsewhere in the notes, supplying any missing proofs or other details is also an exercise for the reader.

Topology

By the traditional epsilon-delta definition, a function f from $\mathbb{R}$ to $\mathbb{R}$ is **continuous** at a point a of $\mathbb{R}$ if

$$\forall\varepsilon\ \exists\delta\ \forall x\ \Big(\varepsilon > 0 \Rightarrow \delta > 0 \land \big(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\big)\Big). \quad (1)$$

We want to simply this, at least theoretically.[1] Given a positive real number r, we define

$$\mathrm{B}(a;r) = \big\{x \in \mathbb{R} \colon |x-a| < r\big\};$$

[1] There is no way to make calculus much easier to learn than it already is. However, different students may find different approaches more congenial. I write the epsilon-delta definition of continuity as in (1), rather than as

$$(\forall\varepsilon > 0)\ (\exists\delta > 0)\ \forall x\ \big(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\big),$$

because in the latter expresion, the first ε is pulled in two directions. Normally $\varepsilon > 0$ and $0 < \varepsilon$ are interchangeable, but we cannot well replace $(\forall\varepsilon > 0)$ with $(\forall 0 < \varepsilon)$.

this is the **(open) ball** in $\mathbb{R}$ with center a and radius r. Now we can rewrite (1) as

$$\forall\varepsilon\ \exists\delta\ \forall x\ \Bigl(\varepsilon > 0 \Rightarrow \delta > 0 \land \Bigl(x \in \mathrm{B}(a;\delta) \Rightarrow f(x) \in \mathrm{B}\bigl(f(a);\varepsilon\bigr)\Bigr)\Bigr). \qquad (2)$$

Using the notation given by

$$f[A] = \{f(x) \colon x \in A\}, \qquad f^{-1}[B] = \{x \colon f(x) \in B\},$$

we can rewrite (2) as

$$\forall\varepsilon\ \exists\delta\ \bigl(\varepsilon > 0 \Rightarrow \delta > 0 \land f[\mathrm{B}(a;\delta)] \subseteq \mathrm{B}(f(a);\varepsilon)\bigr),$$

or else as

$$\forall\varepsilon\ \exists\delta\ \bigl(\varepsilon > 0 \Rightarrow \delta > 0 \land \mathrm{B}(a;\delta) \subseteq f^{-1}[\mathrm{B}(f(a);\varepsilon)]\bigr). \qquad (3)$$

Either of these two formulations is simpler than in (1), because it uses fewer quantifiers. We can eliminate one more quantifier by first defining a **neighborhood** of a point to be a set that includes a ball whose center is the point. Note well the language:[2]

- B **includes** A if $A \subseteq B$;
- B **contains** a if $a \in B$.

A neighborhood of a is now a set N such that, for some radius r, $\mathrm{B}(a;r) \subseteq N$. We can now write (3) as,

For every neighborhood X of $f(a)$,
$f^{-1}[X]$ is a neighborhood of a. (4)

[2]Many writers do not make clear in words the distinction between inclusion and containment.

For the final step in our simplification of (1), we define a set to be **open** if it is a neighborhood of each of its points. We confirm that this definition does not create an ambiguity:

Lemma 1. *Every open ball is an open set.*

Proof. Say $b \in \mathrm{B}(a; r)$. Then $|b - a| < r$. Let $s = r - |b - a|$. Then $\mathrm{B}(b; s) \subseteq \mathrm{B}(a; r)$ since

$$\begin{aligned} |x - b| < s &\implies |x - b| < r - |b - a| \\ &\implies |x - b| + |b - a| < r \\ &\implies |x - a| < r \end{aligned}$$

by the **Triangle Inequality.** □

We finally rewrite (4) as follows.

Theorem 1. *A function is continuous if and only if, under it, the inverse image of every open set is open.*

We have been working in $\mathbb{R}$; but the last theorem will be true by definition in an arbitrary *topological space.*

First, $\mathbb{R}$ is a *metric space,* and every metric space will be a topological space. A **metric** on a set M is a function d from $M \times M$ to $\mathbb{R}$ such that, for all values from M of the variables,

1) $d(x, y) \geqslant 0$,
2) $d(x, y) = 0 \iff x = y$,
3) $d(x, y) = d(y, x)$,
4) $d(x, y) + d(y, z) \geqslant d(x, z)$

(the last condition is the **triangle inequality** for d). The pair (M, d) is then a **metric space.**

Example 1. Let $n \in \mathbb{N}$. Several metrics are definable on $\mathbb{R}^n$:

1) the **Euclidean metric** $(\boldsymbol{x}, \boldsymbol{y}) \mapsto |\boldsymbol{x} - \boldsymbol{y}|$, where[3]

$$|\boldsymbol{z}| = \sqrt{\sum_i} {z_i}^2; \tag{5}$$

2) the **New York metric** $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_i |x_i - y_i|$;
3) $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \max_i |x_i - y_i|$.

The triangle inequality for the Euclidean metric follows from the **Minkowski inequality**

$$|\boldsymbol{x}| + |\boldsymbol{y}| \geqslant |\boldsymbol{x} + \boldsymbol{y}|,$$

which follows from the **Cauchy–Schwartz inequality**

$$|\boldsymbol{x}| \cdot |\boldsymbol{y}| \geqslant \boldsymbol{x} \cdot \boldsymbol{y},$$

where by definition

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_i x_i \cdot y_i.$$

Example 2. If I is a closed, bounded interval of $\mathbb{R}$, and we define

$$\mathrm{C}(I) = \{\text{continuous functions from } I \text{ to } \mathbb{R}\},$$

then $(f, g) \mapsto \int_I |f - g|$ is a metric on $\mathrm{C}(I)$.

Example 3. The **discrete metric** on a set M is

$$(x, y) \mapsto \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

[3] Most writers habitually accompany the radical sign $\sqrt{}$ with a vinculum $\overline{}$. The vinculum is an alternative to parentheses, so that $\sqrt{a+b}$ means $\sqrt{}(a+b)$. However, since multiplication is notationally prior to addition, the expression $\sum_i {z_i}^2$ means not $\left(\sum_i z_i\right)^2$ but $\sum_i({z_i}^2)$, and so $\sqrt{}\sum_i {z_i}^2$ in (5) can only mean $\sqrt{\sum_i {z_i}^2}$, that is, $\sqrt{(\sum_i {z_i}^2)}$.

In any metric space (M, d), if $a \in M$ and r is a positive real number, we define the **(open) ball** with center a and radius r by analogy with the definition in $\mathbb{R}$:

$$\mathrm{B}(a; r) = \{x \in M \colon d(x, a) < r\}.$$

Then neighborhoods and open sets have the same definitions as before, and Lemma 1 is still true, by practically the same proof. Theorem 1 will be true by definition.

To understand the next theorem, one must be clear about the terminology. The **union** of a collection of sets comprises every object that belongs to *some* member of the collection; the **intersection** of the collection comprises every object that belongs to *every* member of the collection.[4] Thus

$$\bigcup \mathscr{A} = \{x \colon \exists Y\ (Y \in \mathscr{A} \land x \in Y)\},$$
$$\bigcap \mathscr{A} = \{x \colon \forall Y\ (Y \in \mathscr{A} \Rightarrow x \in Y)\}.$$

The union of two or more sets is the union of the collection of those sets; likewise for the intersection:

$$A \cup B \cup C \cup \cdots = \bigcup \{A, B, C, \dots\},$$
$$A \cap B \cap C \cap \cdots = \bigcap \{A, B, C, \dots\}.$$

Theorem 2. *In any metric space,*

1) *the union of every nonempty collection of open sets is open,*
2) *the intersection of any two open sets is open,*

[4] A set *comprises* or its elements, and the elements *compose* the set. Some speakers and writers confuse the two verbs. We may also say that a set *consists of* its elements.

3) *the empty set is open,*
4) *the whole space is open.*

The theorem will be true by definition in a topological space. First, we can simplify the statement of the theorem. Obviously

$$\bigcup \varnothing = \varnothing.$$

Also, logically, $\bigcap \varnothing$ consists of everything; but what "everything" means depends on the context. The *complement* of a set comprises everything not in the set; but by "everything" we mean everything in some previously chosen universal set. If we understand the empty set to be a collection (namely the empty collection) of subsets of a universal set Ω, then we may understand

$$\bigcap \varnothing = \Omega.$$

We can understand the last theorem as that, in any metric space,

1) the union of every collection of open sets is open,
2) the intersection of every finite collection of open sets is open.

Exercise 1. Let $f\colon \mathbb{R} \to \mathbb{R}$.

(a) If f is continuous, then the set

$$\Bigl\{\bigl(x, f(x)\bigr) : x \in \mathbb{R}\Bigr\}$$

(namely the **graph** of f) is closed with respect to the Euclidean metric on $\mathbb{R}^2$.

(b) The converse fails.

(c) The graph of f may fail to be closed.

We now define a **topology** on a set as a collection of subsets of the set that contains

1) the union of every collection of its members and
2) the intersection of every finite collection of its members.

If τ is a topology on a set Ω, we call the pair (Ω, τ) a **topological space,** and the elements of τ are the **open subsets** of the space.[5] For a more symbolic presentation, we may define

$$\mathscr{P}(\Omega) = \{\text{subsets of } \Omega\} = \{X \colon X \subseteq \Omega\},$$
$$\mathscr{P}_{\omega}(\Omega) = \{\text{finite subsets of } \Omega\} = \{X \colon X \subseteq \Omega \ \ \& \ \ |X| < \infty\}.$$

Here ω is the set $\{0, 1, 2, \dots\}$ of natural numbers, and this set can be understood as the smallest infinite cardinal number. Thus $\mathscr{P}_{\omega}(\Omega)$ comprises the subsets of Ω that are smaller than ω. By definition then, τ is a topology on Ω if $\tau \subseteq \mathscr{P}(\Omega)$ and

$$\forall \mathscr{X} \ \left(\mathscr{X} \in \mathscr{P}(\tau) \Rightarrow \bigcup \mathscr{X} \in \tau\right),$$
$$\forall \mathscr{X} \ \left(\mathscr{X} \in \mathscr{P}_{\omega}(\tau) \Rightarrow \bigcap \mathscr{X} \in \tau\right).$$

In any topological space, a **neighborhood** of a point is a subset of the space that includes an open set that contains the point. The proof of the next theorem uses the Axiom of Choice.

Theorem 3. *In any topological space, a set is open if and only if it is a neighborhood of each of its points.*

Example 4. Every set with at least two elements can be given two different topologies:

[5] It is an historical peculiarity that many writers use X as a name for an arbitrary topological space, when x has been used since the time of Descartes as a variable.

1) the **discrete topology,** in which every set is open;
2) the **trivial topology,** in which only the empty set and the whole space are open.

Example 5. Every infinite set has a topology that is neither discrete nor trivial: the **cofinite topology,** in which the the empty set and complement of every finite set is open.

In every topological space, the complement of an open set is called **closed.** Thus

1) the intersection of every collection of closed sets is closed,
2) the union of every finite collection of closed sets is closed.

The **closure** of a set is the smallest closed set that includes it. This definition makes sense, precisely because the intersection of every collection of closed sets is closed, and every member of the collection includes the intersection. In a topological space (Ω, τ), we may write

$$X^{\mathrm{c}} = \Omega \smallsetminus X = \{x \in \Omega \colon x \notin X\}.$$

Then, for the closure of a set E, we write

$$\overline{E} = \bigcap \{X \colon X^{\mathrm{c}} \in \tau \ \ \& \ \ E \subseteq X\}.$$

Theorem 4. *In any topological space Ω, closure is an operation φ on $\mathscr{P}(\Omega)$ satisfying*

1) $X \subseteq \varphi(X)$,
2) $\varphi(\varphi(X)) = \varphi(X)$,
3) $\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y)$,
4) $\varphi(\varnothing) = \varnothing$.

Conversely, if some operation φ on $\mathscr{P}(\Omega)$ has these properties, then φ is the closure operation for a topology on Ω.

Example 6. The operation

$$X \mapsto \begin{cases} X, & \text{if } |X| < \infty, \\ \Omega, & \text{if } |X| = \infty, \end{cases}$$

satisfies the conditions of the theorem and is therefore the closure operation for a topology on Ω. This topology is the cofinite topology.

Exercise 2. In a metric space (M, d),

(a) $\overline{X} = \big\{y \in M \colon \inf\{d(y, z) \colon z \in X\} = 0\big\}$,

(b) $\overline{\mathrm{B}(a; r)} \subseteq \{x \in M \colon d(a, x) \leqslant r\}$, but

(c) the last inclusion can be strict.

Example 7. Always

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B};$$

but in $\mathbb{R}$ with the usual Euclidean topology,

$$\overline{\mathbb{Q}} = \mathbb{R}, \qquad \overline{\mathbb{Q}^c} = \mathbb{R}, \qquad \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R}, \qquad \overline{\mathbb{Q} \cap \mathbb{Q}^c} = \varnothing.$$

Exercise 3. Let $B \subseteq \Omega$.

(a) Show that the function

$$X \mapsto \begin{cases} X \cup B, & \text{if } X \neq \varnothing, \\ \varnothing, & \text{if } X = \varnothing, \end{cases}$$

is the closure operation for a topology on Ω.

(b) What are the open sets in this topology?

The **interior** of a set is the largest open set that it includes. The definition makes sense, like the definition of the closure of a set. In a topological space (Ω, τ), we may write

$$E^{\circ} = \bigcap \{X \colon X \in \tau \ \& \ X \subseteq E\}.$$

Theorem 5. *The interior of a set consists precisely of the points of which the set is a neighborhood.*

Proof. For all subsets E of a topological space, for all points a of the space, the following statements are equivalent:

1. E is a neighborhood of a.
2. $a \in O$ and $O \subseteq E$, for some open set O.
3. $a \in E^\circ$. □

Theorem 6. *For all subsets E of a topological space,*

$$(E^\circ)^{\mathrm{c}} = \overline{E^{\mathrm{c}}}, \tag{6}$$

$$E^\circ = \overline{E^{\mathrm{c}}}^{\mathrm{c}}, \tag{7}$$

$$(E^{\mathrm{c}})^\circ = \overline{E}^{\mathrm{c}}. \tag{8}$$

Proof. If the topology is τ, we compute

$$\begin{aligned}(E^\circ)^{\mathrm{c}} &= \Big(\bigcup\{X \in \tau \colon X \subseteq E\}\Big)^{\mathrm{c}} \\ &= \bigcap\{X^{\mathrm{c}} \colon X \in \tau \ \& \ X \subseteq E\} \\ &= \bigcap\{X^{\mathrm{c}} \colon X \in \tau \ \& \ E^{\mathrm{c}} \subseteq X^{\mathrm{c}}\} \\ &= \bigcap\{X \colon X^{\mathrm{c}} \in \tau \ \& \ E^{\mathrm{c}} \subseteq X\} = \overline{E^{\mathrm{c}}},\end{aligned}$$

so we have (6) and then (7). Replacing E with E^{c} yields (8). □

Exercise 4. (a) Show that

$$\overline{X^\circ} = \overline{\overline{X^\circ}^\circ}. \tag{9}$$

(b) Conclude that

$$\overline{\overline{X}^{\mathrm{c}}} = \overline{\overline{\overline{\overline{X}^{\mathrm{c}}}^{\mathrm{c}}}^{\mathrm{c}}}, \tag{10}$$

and therefore the smallest collection of subsets of a topological space that contains a given set and is closed under the operations of taking closures, complements, and interiors has no more than 14 members.

(c) If the space is $\mathbb{R}$, there is an example with exactly 14 members.

A **limit point** of a subset of a topological space is a point whose every neighborhood contains an element of the subset other than the point itself. The set of all limit points of a set A can be denoted by

$$A';$$

this is the **Cantor–Bendixson derivative** of A.

Theorem 7. $\overline{A} = A \cup A'$.

Proof. By Theorem 6 we have

$$\begin{aligned}
x \in \overline{A} &\iff x \notin (A^{\mathrm{c}})^{\circ} \\
&\iff A^{\mathrm{c}} \text{ is not a neighborhood of } x \\
&\iff \text{every neighborhood of } x \text{ contains a point of } A \\
&\iff x \in A \text{ or } x \in A' \\
&\iff x \in A \cup A'.
\end{aligned}$$

□

Example 8. In $\mathbb{R}$, we have

$$\mathbb{Q}' = \mathbb{R}, \qquad \left\{\frac{1}{n} \colon n \in \mathbb{N}\right\} = \{0\}, \qquad \{0\}' = \varnothing;$$

in $\mathbb{R}^2$, if $A = \{(1/m, 1/n) \colon (m, n) \in \mathbb{N}^2\}$, then

$$A' = \big\{(0,0)\big\} \cup \left\{\left(\frac{1}{m}, 0\right) : m \in \mathbb{N}\right\} \cup \left\{\left(0, \frac{1}{n}\right) : n \in \mathbb{N}\right\},$$
$$A'' = \big\{(0,0)\big\},$$
$$A''' = \varnothing.$$

Example 9. Letting Ω be the subset $\{0\} \cup \{p\colon p \text{ prime}\}$ of $\mathbb{Z}$, we are going to define a topology on Ω so that

$$\overline{\{0\}} = \Omega, \qquad \overline{\{p\}} = \{p\}, \qquad \Omega' = \Omega. \tag{11}$$

The topology arises naturally as follows. If $a \in \mathbb{Z}$, we define

$$(a) = \{ax\colon x \in \mathbb{Z}\}.$$

If also $b \in \mathbb{Z}$, we define

$$(a, b) = \{ax + by\colon (x, y) \in \mathbb{Z}^2\},$$

and then

$$\gcd(a, b) = \begin{cases} \min\bigl(\mathbb{N} \cap (a, b)\bigr), & \text{if one of } a \text{ and } b \text{ is not } 0, \\ 0, & \text{if both } a \text{ and } b \text{ are } 0. \end{cases}$$

Consequently

$$(a, b) = \bigl(\gcd(a, b)\bigr).$$

The subsets (a) and (a, b) are **ideals** of $\mathbb{Z}$ (because they are additive subgroups of $\mathbb{Z}$ that are closed under multiplication by arbitrary elements of $\mathbb{Z}$). A *proper* ideal I of $\mathbb{Z}$ is called **prime** if

$$xy \in I \land x \notin I \implies y \in I.$$

Then the prime ideals of $\mathbb{Z}$ are precisely (p), where p is a prime number, and also (0). This is by Euclid's Lemma, whereby

$$p \mid ab \land p \nmid a \implies p \mid b;$$

also

$$ab = 0 \land a \neq 0 \implies b = 0.$$

Since (1) is an improper ideal, it is not prime. We denote the set of prime ideals of $\mathbb{Z}$ by

$$\operatorname{Spec}(\mathbb{Z});$$

this is the **spectrum** of $\mathbb{Z}$, and we topologize it by defining

$$\mathrm{V}(a) = \{I \in \operatorname{Spec}(\mathbb{Z}) \colon a \in I\}.$$

Then

$$\mathrm{V}(a) = \begin{cases} \bigl\{(p) \colon p \mid a\bigr\}, & \text{if } a \neq 0, \\ \operatorname{Spec}(\mathbb{Z}), & \text{if } a = 0. \end{cases}$$

In particular, $\mathrm{V}(a)$ is finite when $a \neq 0$. Moreover, for any *nonempty* set P of prime numbers,

$$\bigl\{(p) \colon p \in P\bigr\} = \mathrm{V}\left(\prod P\right).$$

Here $\prod P$ is the product of the elements of P. By definition, $\prod \varnothing = 1$, and $\mathrm{V}(1) = \varnothing$. Then the sets $\mathrm{V}(a)$ are the closed sets of a topology on $\operatorname{Spec}(\mathbb{Z})$. A way to confirm this is to note first

$$\mathrm{V}(a) \cup \mathrm{V}(b) = \mathrm{V}(ab).$$

We can define $\gcd(a, b, c)$ and so forth in the obvious way, and if $A \subseteq \mathbb{Z}$, we can define

$$\gcd(A) = \min\bigl\{\gcd(X) \colon X \in \mathscr{P}_\omega(A)\bigr\}.$$

Then

$$\bigcap_{x \in A} \mathrm{V}(x) = \mathrm{V}\bigl(\gcd(A)\bigr).$$

This is still true when A is empty, if we understand $\gcd(\varnothing)$ to be 0. That is, $\operatorname{Spec}(\mathbb{Z}) = \mathrm{V}(0)$. Also $\mathrm{V}(1) = \varnothing$. Consequently the sets $\mathrm{V}(a)$ are the closed sets of a topology on $\mathrm{Spec}(\mathbb{Z})$, called the **Zariski topology.** Such a topology is defined on the spectrum of prime ideals of any commutative ring. When the ring is $\mathbb{Z}$, the Zariski topology is like the cofinite topology, except that (0) belongs to every nonempty open set. Consequently, if we confuse Ω as above with $\mathrm{Spec}(\mathbb{Z})$ in the obvious way, we have (11).

Example 10. If again Ω is the subset $\{0\} \cup \{p\colon p \text{ prime}\}$ of $\mathbb{Z}$, we shall define a topology on Ω so that

$$\overline{\{0\}} = \{0\}, \qquad \overline{\{p\}} = \{p\}, \qquad \Omega' = \{0\}. \tag{12}$$

Here Ω is precisely the set of possible **characteristics** of fields. The characteristic of an arbitrary commutative ring is the least n in $\mathbb{N}$, if there is one, such that $n \cdot 1 = 0$ in the ring; otherwise the characteristic is 0. The field $\mathbb{F}_p$ with p elements is the smallest field of characteristic p; the field $\mathbb{Q}$ of rational numbers is the smallest field of characteristic p. Using the next theorem, we give the set $\{\mathbb{Q}\} \cup \{\mathbb{F}_p\colon p \text{ prime}\}$ the *coarsest* topology in which the sets $\{K\colon \mathrm{char}(K) = p\}$ and their complements are open. Then every one-element set is its own closure, but $\{\mathbb{Q}\}$ is the only limit point of the space.

One topology on a set is **coarser** than another, and the latter is **finer** than the former, if the latter includes the former.

Theorem 8. *For any set Ω, for any subset $\mathscr{S}$ of $\mathscr{P}(\Omega)$, if we let*

$$\mathscr{B} = \left\{\bigcap Y : Y \in \mathscr{P}_\omega(S)\right\},$$

then the set

$$\left\{\bigcup \mathscr{X} : \mathscr{X} \in \mathscr{P}(\mathscr{B})\right\}$$

is a topology on Ω and is the coarsest topology on Ω that includes $\mathscr{S}$.

Proof. Let the set in question be called $\mathscr{T}$. If τ is a topology on Ω that includes $\mathscr{S}$, then $\mathscr{B} \subseteq \tau$, and then τ must also include $\mathscr{T}$. Suppose $\mathscr{A}$ and $\mathscr{C}$ are subsets of $\mathscr{B}$. Then

$$\bigcup \mathscr{A} \cap \bigcup \mathscr{C} = \bigcup \{X \cap Y \colon X \in \mathscr{A} \ \& \ Y \in \mathscr{C}\},$$

which is in $\mathscr{T}$. Since also the empty set is in $\mathscr{T}$, this is closed under finite intersections. Finally, suppose $\boldsymbol{A}$ is a collection of subsets of $\mathscr{B}$. Then $\bigcup \boldsymbol{A}$ is a subset of this, and

$$\begin{aligned}
&\bigcup \Big\{ \bigcup \mathscr{X} \colon \mathscr{X} \in \boldsymbol{A} \Big\} \\
&= \Big\{ z \colon \exists \mathscr{X} \ \Big(\mathscr{X} \in \boldsymbol{A} \ \& \ z \in \bigcup \mathscr{X} \Big) \Big\} \\
&= \{ z \colon \exists \mathscr{X} \ \exists Y \ (\mathscr{X} \in \boldsymbol{A} \ \& \ Y \in \mathscr{X} \ \& \ z \in Y) \} \\
&= \{ z \colon \exists Y \ \exists \mathscr{X} \ (\mathscr{X} \in \boldsymbol{A} \ \& \ Y \in \mathscr{X} \ \& \ z \in Y) \} \\
&= \Big\{ z \colon \exists Y \ \Big(Y \in \bigcup \boldsymbol{A} \ \& \ z \in Y \Big) \Big\} \\
&= \bigcup \bigcup \boldsymbol{A},
\end{aligned}$$

which is therefore in $\mathscr{T}$. Thus $\mathscr{T}$ is closed under arbitrary unions. □

In the theorem, $\mathscr{B}$ is a **base** for the topology $\mathscr{T}$, because every member of $\mathscr{T}$ is a union of members of $\mathscr{B}$. Also $\mathscr{S}$ is a **sub-base** for $\mathscr{T}$. By the theorem, every set is a sub-base for some topology.

Theorem 9. *For any set Ω, a subset $\mathscr{B}$ of $\mathscr{P}(\Omega)$ is a base for a topology on Ω if and only if $\bigcup \mathscr{B} = \Omega$ and, for any two elements A and C of $\mathscr{B}$, for any d in $A \cap C$, for some E in $\mathscr{B}$,*

$$E \subseteq A \cap C, \qquad d \in E.$$

Exercise 5. Show that the collection of open squares

$$(a, a+\delta) \times (b, b+\delta)$$

is a base for a topology on $\mathbb{R}^2$.

Theorem 10. *For any topological space (Ω, τ), for any subset A of Ω, the set*

$$\{X \cap A \colon X \subseteq \Omega\}$$

is a topology on A. A subset X of A is closed in this topology if and only if, for some closed subset F of Ω,

$$X = A \cap F.$$

In any case, if $\overline{X}$ is the closure of X in Ω, then $A \cap \overline{X}$ is its closure in A.

In the theorem, A with its topology is a **subspace** of (Ω, τ).

As noted earlier, a function f from a topological space A to a topological space C is **continuous** if $f^{-1}[U]$ is open in A for every open subset U of C, that is, the inverse image of every open set is open. Since

$$f^{-1}[X]^{\mathrm{c}} = f^{-1}[X^{\mathrm{c}}],$$

the function f is continuous if and only if the inverse image of every closed set is closed. If B is a subspace of C and $f[A] \subseteq B$, then f is still continuous as a function from A to B. If $f[A] = B$, and f is injective, and f^{-1} is continuous, then f is called a **homeomorphism** from A to B, and the spaces A and B are **homeomorphic** to one another.

Example 11. All open intervals of $\mathbb{R}$ are homeomorphic to one another.

If A and B are two topological spaces, then $A \times B$ has a topologically consisting of the products of open subsets of A and B.

Exercise 6. The product topology on $\mathbb{R} \times \mathbb{R}$ is the Euclidean topology on $\mathbb{R}^2$.

If we are given an indexed family $(\Omega_i : i \in I)$ of topological spaces, we define its **product** by the identity

$$\prod_{i \in I} \Omega_i = \{(x_i : i \in I) : \forall i \; (i \in I \Rightarrow x_i \in \Omega_i)\}.$$

Here $(x_i : i \in I)$ is just the function $i \mapsto x_i$ having domain Ω_i. (Its range is a subset of $\bigcup_{i \in I} \Omega_i$.) For each j in I, there is a **projection** π_j from $\prod_{i \in I} \Omega_i$ onto Ω_j given by

$$\pi_j(x_i : i \in I) = x_j.$$

Then the **product topology** or **Tychonoff topology** on $\prod_{i \in I} \Omega_i$ is the coarsest in which the projections are continuous. This topology has a sub-base consisting of the sets ${\pi_j}^{-1}[U_j]$, where $j \in I$ and $U_j \in \tau_j$. Then it has a base consisting of the sets

$$\bigcap_{j \in J} {\pi_j}^{-1}[U_j],$$

where J is a finite subset of I. The same sub-basic sets can be written as $\prod_{i \in I} \Omega_i^*$, where

$$\Omega_i^* = \begin{cases} U_i, & \text{if } i \in J, \\ \Omega_i, & \text{if } i \in I \smallsetminus J. \end{cases}$$

In case all of the spaces Ω_i are the same space Ω, we write $\prod_{i \in I} \Omega_i$ as Ω^I: this is the space of functions from I to Ω.

Example 12. The simplest nontrivial example of an infinite product of spaces is 2^ω, where $2 = \{0, 1\}$. We have a bijection

$$f \mapsto \{i \in \omega \colon f(i) = 1\} \tag{13}$$

from 2^ω to $\mathscr{P}(\omega)$. In the Tychonoff topology, 2^ω has a basis consisting of the sets

$$\left\{f \in 2^\omega \colon \bigwedge_{i<n} f(i) = e_i\right\},$$

where $n \in \omega$ and $(e_i \colon i < n) \in 2^n$. Here n is understood as the set $\{0, \dots, n-1\}$ or $\{i \colon i < n\}$. We may then speak of the **Tychonoff topology** on $\mathscr{P}(\omega)$, with basis consisting of the sets

$$\left\{X \subseteq \omega \colon \bigwedge_{i<n} (i \in X \Leftrightarrow e_i = 1)\right\}.$$

Exercise 7. (a) A metric can be defined on $\mathscr{P}(\omega)$ by the rule

$$d(X, Y) = \sum_{i \in X \triangle Y} \frac{1}{2^i},$$

where

$$X \triangle Y = (X \smallsetminus Y) \cup (Y \smallsetminus X) = (X \cup Y) \smallsetminus (X \cap Y).$$

(b) The induced topology on $\mathscr{P}(\omega)$ makes the bijection (13) from 2^ω to $\mathscr{P}(\omega)$ a homeomorphism.

(c) The function

$$X \mapsto \sum_{i \in X} \frac{2}{3^{i+1}}$$

from $\mathscr{P}(\omega)$ to $\mathbb{R}$ is a homeomorphism onto its image, which is the **Cantor set.**

The **Cantor Intersection Theorem** is that any decreasing sequence of closed bounded subsets of $\mathbb{R}^n$ has nonempty intersection. That is, if

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots,$$

where F_0 is bounded, and each F_n is closed, then

$$\bigcap_{n\in\omega} F_n \neq \varnothing.$$

We prove this in $\mathbb{R}$. Note that the F_n must be both closed and bounded, since

$$\bigcap_{n\in\omega}[n,\infty) = \varnothing, \qquad \bigcap_{n\in\omega}\left(0,\frac{1}{n+1}\right) = \varnothing.$$

But suppose the F_n are bounded. Let

$$a_n = \inf F_n.$$

Then the sequence $(a_n : n \in \omega)$ is increasing and bounded, so it has a limit b (namely $\sup_{n\in\omega} a_n$). Since the F_n are closed, $a_n \in F_n$. Then for all k in ω, $a_{n+k} \in F_n$. If the set $\{a_n : n \in \omega\}$ is finite, then b belongs to it, and therefore

$$b \in \bigcap_{n\in\omega} F_n.$$

If $\{a_n : n \in \omega\}$ is finite, then b is a limit point of it, and so b is a limit point of each F_n. Therefore, by Theorem 7, b belongs to each F_n, since this is closed.

The explanation for why the theorem is correct is that closed bounded subsets of R^n are *compact.* We shall see similarly that, in the Tychonoff topology, $\mathscr{P}(\omega)$ is compact.

If G is a group, then $\mathscr{P}(G)$ can be given the Tychonoff topology. Here the subset $\{H \in \mathscr{P}(G) \colon H \leqslant G\}$ consisting of subgroups of G is closed, since it is

$$\{X \in \mathscr{P}(G) \colon \mathrm{e} \in X\} \cap \bigcap_{(a,b)\in G^2} \{X \in \mathscr{P}(G) \colon a \in G \land b \in G \Rightarrow ab^{-1} \in G\},$$

and

$$\{X \colon a \in G \land b \in G \Rightarrow ab^{-1} \in G\} = \{X \colon a \notin G\} \cup \{X \colon b \notin G\} \cup \{X \colon ab^{-1} \in G\}.$$

We shall see that therefore $\{H \in \mathscr{P}(G) \colon H \leqslant G\}$ is compact. Similarly the set of normal subgroups of G is closed in $\mathscr{P}(G)$ and is therefore compact. If G is a finitely generated free group, then a quotient G/N of G is a *limit group* if, in the Tychonoff topology, N is the limit of a sequence $(N_k \colon k \in \omega)$ of normal subgroups of G such that each quotient G/N_k is free.

The Tychonoff topology on $\mathscr{P}(\omega)$ arises in propositional logic. Here we start with a set $\{P_n \colon n \in \omega\}$ of **propositional variables.** Each of these is also a **propositional formula,** and if F and G are propositional formulas, then so are $\neg F$ and $(F \land G)$. Every propositional formula F determines a function $\widehat{F}$ from $\mathscr{P}(\omega)$ to 2 so that

$$\widehat{P_n}(X) = 1 \iff n \in X,$$
$$\widehat{\neg F}(X) = \widehat{F}(X) + 1,$$
$$(\widehat{F \land G})(X) = \widehat{F}(X) \cdot \widehat{G}(X),$$

where the operations are as in $\mathbb{Z}_2$, that is, $\mathbb{F}_2$. If $\widehat{F}(X) = 1$, we may say that F is **true** in X, or X is a **model** of F. We define

$$\mathrm{Mod}(F) = \{X \in \mathscr{P}(\omega) \colon \widehat{F}(X) = 1\};$$

its elements are just the models of $\mathscr{F}$. If $\mathscr{F}$ is a collection of propositional formulas, we define

$$\mathrm{Mod}(\mathscr{F}) = \bigcap_{F \in \mathscr{F}} \mathrm{Mod}(F).$$

These sets $\mathrm{Mod}(\mathscr{F})$ are precisely the closed subsets of $\mathscr{P}(\omega)$ in the Tychonoff topology. Suppose $\mathscr{F} = \{F_n \colon n \in \omega\}$, and let

$$A_n = \mathrm{Mod}(F_k \colon k < n).$$

Then $(A_n \colon n \in \omega)$ is a decreasing sequence of closed subsets of $\mathscr{P}(\omega)$. If each A_n is nonempty, we shall show that their intersection is nonempty. This means $\mathscr{F}$ has a model, provided that every finite subset of $\mathscr{F}$ has a model.

We use logic similarly to obtain the topology of Example 10. Now let $\boldsymbol{C}$ be the class of all fields. (It is actually a proper class, not a set.) If σ is a sentence of the language of field theory, and $K \in \boldsymbol{C}$, we write

$$K \models \sigma$$

whenever σ is true in K. For example,

$$\mathbb{Q} \models \neg\exists x\ x^2 = 2, \qquad \mathbb{R} \models \exists x\ x^2 = 2.$$

Now let

$$\mathbf{Mod}(\sigma) = \{K \in \boldsymbol{C} \colon K \models \sigma\}.$$

Then for all primes p,

$$\{K \in \boldsymbol{C} \colon \operatorname{char}(K) = p\} = \mathbf{Mod}(p \cdot 1 = 0),$$

where $p \cdot 1$ means $1 + \cdots + 1$, with p copies of 1. We consider the topology with sub-base consisting of the classes $\mathbf{Mod}(p \cdot 1 = 0)$ and their complements. In this topology, if two fields have the same characteristic, then they belong to the same open classes. So we might as well consider the class of all fields having a given characteristic as a single point. This is practically what we did in Example 10, where we considered the subspace $\{\mathbb{Q}\} \cup \{\mathbb{F}_p \colon p \text{ prime}\}$ or Ω, with one field of each characteristic. The function

$$K \mapsto \begin{cases} \mathbb{Q}, & \text{if } \operatorname{char}(K) = 0, \\ \mathbb{F}_p, & \text{if } \operatorname{char}(K) = p \end{cases}$$

from $\boldsymbol{C}$ to Ω is continuous, and it induces a well-defined homeomorphism from $\boldsymbol{C}/{\sim}$ to Ω, where

$$K \sim L \iff \operatorname{char}(K) = \operatorname{char}(L),$$

and $\boldsymbol{C}/{\sim}$ is given the *quotient topology.*

In general, for an arbitrary topological space Ω, if $\sim$ is an equivalence relation on Ω, then the **quotient topology** on $\Omega/{\sim}$ is the finest in which the quotient map $x \mapsto [x]$ from Ω to $\Omega/{\sim}$ is continuous.

Example 13. The **torus** is $\mathbb{R}^2/{\sim}$, where

$$(a, b) \sim (x, y) \iff (a - x, b - y) \in \mathbb{Z}^2.$$

Exercise 8. A **Gromov–Hausdorff metric** on $\mathscr{P}(\omega)$ is given by the rule

$$d(X, Y) = \frac{1}{2^{\min(X \triangle Y)}}.$$

Show that this is indeed a metric, giving the Tychonoff topology as before.