Topology

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These notes are based on lectures in a graduate course of topology given in the fall semester of 2016–7. The notes are not guaranteed to be free of error. I may edit them in future, or add to them. There are explicitly labelled exercises, to be solved as homework. Grammatically, an exercise can be a question, a command, or a statement; it is then to be answered, obeyed, or proved, respectively. Elsewhere, supplying any missing proofs or other details (as well as correcting inadvertent mistakes) is also an exercise for the reader.

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1 Continuity

By the traditional epsilon-delta definition, a function f from $\mathbb{R}$ to $\mathbb{R}$ is **continuous** at a point a of $\mathbb{R}$ if

$$\forall\varepsilon\ \exists\delta\ \forall x\ \Bigl(\varepsilon > 0 \Rightarrow \delta > 0 \land \bigl(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\bigr)\Bigr). \tag{1}$$

We want to simply this, at least theoretically.[1] Given a positive real number r, we define

$$\mathrm{B}(a;r) = \bigl\{x \in \mathbb{R}\colon |x-a| < r\bigr\}; \tag{2}$$

this is the **(open) ball** in $\mathbb{R}$ with center a and radius r. Now we can rewrite (1) as

$$\forall\varepsilon\ \exists\delta\ \forall x\ \Bigl(\varepsilon > 0 \Rightarrow \delta > 0 \land \Bigl(x \in \mathrm{B}(a;\delta) \Rightarrow f(x) \in \mathrm{B}\bigl(f(a);\varepsilon\bigr)\Bigr)\Bigr). \tag{3}$$

Using the notation given by

$$f[A] = \{f(x)\colon x \in A\}, \qquad f^{-1}[B] = \{x\colon f(x) \in B\},$$

[1] There is no way to make calculus much easier to learn than it already is. However, different students may find different approaches more congenial. I write the epsilon-delta definition of continuity as in (1), rather than as

$$(\forall\varepsilon > 0)\ (\exists\delta > 0)\ \forall x\ \bigl(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\bigr),$$

because in the latter expresion, the first ε is pulled in two directions. Normally $\varepsilon > 0$ and $0 < \varepsilon$ are interchangeable, but we cannot well replace $(\forall\varepsilon > 0)$ with $(\forall 0 < \varepsilon)$.

we can rewrite (3) as

$$\forall\varepsilon\ \exists\delta\ \Big(\varepsilon > 0 \Rightarrow \delta > 0 \wedge f\big[\mathrm{B}(a;\delta)\big] \subseteq \mathrm{B}\big(f(a);\varepsilon\big)\Big), \tag{4}$$

or else as

$$\forall\varepsilon\ \exists\delta\ \Bigg(\varepsilon > 0 \Rightarrow \delta > 0 \wedge \mathrm{B}(a;\delta) \subseteq f^{-1}\Big[\mathrm{B}\big(f(a);\varepsilon\big)\Big]\Bigg). \tag{5}$$

Either of (4) and (5) is simpler than (1), because it uses fewer quantifiers. We can eliminate one more quantifier by first defining a **neighborhood** of a point to be a set that includes a ball whose center is the point. Note well the language:[2]

- B **includes** A if $A \subseteq B$;
- B **contains** a if $a \in B$.

A neighborhood of a is now a set N such that, for some radius r, $\mathrm{B}(a;r) \subseteq N$. We can now write (5) as,

For every neighborhood X of $f(a)$,
$f^{-1}[X]$ is a neighborhood of a. (6)

For the final step in our simplification of (1), we define a set to be **open** if it is a neighborhood of each of its points. We confirm that this definition does not create an ambiguity:

Theorem 1. *Every open ball is an open set.*

Proof. Say $b \in \mathrm{B}(a;r)$. Then $|b-a| < r$. Let $s = r - |b-a|$. Then $\mathrm{B}(b;s) \subseteq \mathrm{B}(a;r)$ since, by the **Triangle Inequality,**

$$\begin{aligned} |x-b| < s &\implies |x-b| < r - |b-a| \\ &\implies |x-b| + |b-a| < r \\ &\implies |x-a| < r. \end{aligned}$$ □

[2]Many writers do not make clear in words the distinction between inclusion and containment.

We now obtain from (6) a characterization of continuity, simply (that is, continuity at all points of a domain):

Theorem 2. *A function from $\mathbb{R}$ to $\mathbb{R}$ is continuous if and only if, under it, the inverse image of every open set is open.*

2 Metric spaces

Theorem 2 will be true by definition if the two instances of $\mathbb{R}$ are replaced with arbitrary *topological spaces,* possibly different.

We first observe that $\mathbb{R}$ is a *metric space,* and every metric space will be a topological space. A **metric** on a set M is a function d from $M \times M$ to $\mathbb{R}$ such that, for all values from M of the variables,

1) $d(x, y) \geqslant 0$,
2) $d(x, y) = 0 \Leftrightarrow x = y$,
3) $d(x, y) = d(y, x)$,
4) $d(x, y) + d(y, z) \geqslant d(x, z)$.

The last condition is the **triangle inequality** for d. The pair (M, d) is a **metric space.** We may that M is a metric space, if d can be understood.

We let $\mathbb{N}$ denote the set $\{1, 2, 3, \dots\}$ of **counting numbers.**

Example 1. Let $n \in \mathbb{N}$. Several metrics are definable on $\mathbb{R}^n$:

1) the **Euclidean metric** $(\boldsymbol{x}, \boldsymbol{y}) \mapsto |\boldsymbol{x} - \boldsymbol{y}|$, where[3]

$$|\boldsymbol{z}| = \sqrt{\sum_i} z_i{}^2; \tag{7}$$

2) the **New-York metric** $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_i |x_i - y_i|$;
3) $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \max_i |x_i - y_i|$.

The triangle inequality for the Euclidean metric follows from the **Minkowski inequality**

$$|\boldsymbol{x}| + |\boldsymbol{y}| \geqslant |\boldsymbol{x} + \boldsymbol{y}|,$$

which follows from the **Cauchy–Schwartz inequality**

$$|\boldsymbol{x}| \cdot |\boldsymbol{y}| \geqslant \boldsymbol{x} \cdot \boldsymbol{y},$$

where by definition

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_i x_i \cdot y_i.$$

Example 2. If I is a closed, bounded interval of $\mathbb{R}$, then

$$(f, g) \mapsto \int_I |f - g| \tag{8}$$

is a metric on the set of continuous functions from I to $\mathbb{R}$.

[3] Most writers habitually combine the radical sign $\surd$ with a *vinculum,* an overline. The vinculum is an alternative to parentheses, so that $\sqrt{a+b}$ means $\surd(a+b)$. Often parentheses would not be needed; this means the vinculum is not needed either. Since multiplication is notationally prior to addition, the expression $\sum_i z_i{}^2$ means not $(\sum_i z_i)^2$ but $\sum_i (z_i{}^2)$. Thus $\sqrt{\sum_i} z_i{}^2$ in (7) can only mean $\sqrt{\sum_i (z_i{}^2)}$.

Example 3. The **discrete metric** on a set M is

$$(x,y) \mapsto \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

In this metric, all triangles are equilateral.

Example 4 (The p-adic metric)**.** For every prime number p, for every nonzero element a of $\mathbb{Q}$, there is a unique element n of $\mathbb{Z}$ such that, for some k and m in $\mathbb{Z}$, neither of which is divisible by p,

$$a = p^n \cdot \frac{k}{m}.$$

In this case we define

$$\mathrm{v}_p(a) = n.$$

For the moment, we also let a' be such that $a = p^{\mathrm{v}_p(a)} \cdot a'$. If also $b \in \mathbb{Q} \smallsetminus \{0\}$, and $\mathrm{v}_p(a) \leqslant \mathrm{v}_p(b)$, then

$$a \pm b = p^{\mathrm{v}_p(a)} \cdot (a' \pm p^{\mathrm{v}_p(b) - \mathrm{v}_p(a)} \cdot b'),$$

so that

$$\mathrm{v}_p(a \pm b) \begin{cases} = \mathrm{v}_p(a), & \text{if } \mathrm{v}_p(a) < \mathrm{v}_p(b), \\ \geqslant \mathrm{v}_p(a), & \text{if } \mathrm{v}_p(a) = \mathrm{v}_p(b). \end{cases}$$

We define

$$\mathrm{v}_p(0) = \infty.$$

Then for all a and b in $\mathbb{Q}$,

$$\mathrm{v}_p(a \pm b) \geqslant \min(\mathrm{v}_p(a), \mathrm{v}_p(b)).$$

This *implies* equality in case $\mathrm{v}_p(a) \neq \mathrm{v}_p(b)$, since if we have $\mathrm{v}_p(a) < \mathrm{v}_p(b)$, then

$$\begin{gathered}\mathrm{v}_p(a+b) \geqslant \mathrm{v}_p(a) = \mathrm{v}_p(a+b-b) \geqslant \min(\mathrm{v}_p(a+b), \mathrm{v}_p(b)),\\ \min(\mathrm{v}_p(a+b), \mathrm{v}_p(b)) = \mathrm{v}_p(a+b),\\ \mathrm{v}_p(a+b) = \mathrm{v}_p(a),\end{gathered}$$

and likewise for $\mathrm{v}_p(a-b)$. Finally, we define

$$|a|_p = \begin{cases} p^{-\mathrm{v}_p(a)}, & \text{if } a \neq 0,\\ 0, & \text{if } a = 0.\end{cases}$$

Then the function $(x, y) \mapsto |x-y|_p$ is a metric on $\mathbb{Q}$, called the ***p*-adic metric.** In particular, the triangle inequality follows from the stronger inequality

$$|x - z|_p \leqslant \max(|x-y|_p, |y-z|_p).$$

Because this stronger rule is satisfied, the p-adic metric is called an **ultrametric.** In such a metric, all triangles are isosceles or equilateral, and in the isosceles case, the third side is shorter than the two equal sides.

In any metric space (M, d), if $a \in M$ and r is a positive real number, we define the **(open) ball** with center a and radius r by analogy with the definition (2) in $\mathbb{R}$:

$$\mathrm{B}(a; r) = \{x \in M \colon d(x, a) < r\}.$$

Then **neighborhoods** and **open sets** have the same definitions as before, and Theorem 1 is still true, by practically the same proof. Theorem 2 will be true by definition.

To understand the next theorem, one must be clear about the terminology. The **union** of a collection of sets comprises every object that belongs to *some* member of the collection; the **intersection** of the collection comprises every object that belongs to *every* member of the collection.[4] Thus

$$\bigcup \mathscr{A} = \{x \colon \exists Y\ (Y \in \mathscr{A} \land x \in Y)\},$$
$$\bigcap \mathscr{A} = \{x \colon \forall Y\ (Y \in \mathscr{A} \Rightarrow x \in Y)\}.$$

The union of two or more sets is the union of the collection of those sets; likewise for the intersection:

$$A \cup B \cup C \cup \dots \cup Z = \bigcup\{A, B, C, \dots, Z\},$$
$$A \cap B \cap C \cap \dots \cap Z = \bigcap\{A, B, C, \dots, Z\}.$$

If $\mathscr{A} = \{X_i \colon i \in I\}$, then we may use the notation

$$\bigcup \mathscr{A} = \bigcup_{i \in I} X_i, \qquad \bigcap \mathscr{A} = \bigcap_{i \in I} X_i.$$

Theorem 3. *In every metric space,*

1) *the union of every nonempty collection of open sets is open,*
2) *the intersection of any two open sets is open,*
3) *the empty set is open,*
4) *the whole space is open.*

The theorem will be true by definition in a topological space. First, we can simplify the statement of the theorem. Obviously

$$\bigcup \varnothing = \varnothing.$$

[4]A set *comprises* its elements, and the elements *compose* the set. Some speakers and writers confuse the two verbs. We may also say that a set *consists of* its elements.

If the intersection of any two open sets is open, then, by induction, the intersection of any finite nonzero number of open sets is open. Logically, $\bigcap \varnothing$ consists of everything; but what "everything" means depends on the context. The *complement* of a set comprises everything not in the set; but by "everything" we mean everything in some previously chosen universal set. If we understand the empty set to be a collection (namely the empty collection) of subsets of a universal set Ω, then we may understand

$$\bigcap \varnothing = \Omega.$$

We can understand the last theorem as that, in any metric space,

1) the union of every collection (including the empty collection) of open sets is open,
2) the intersection of every finite collection (including the empty collection) of open sets is open.

Exercise 1. Let $f \colon \mathbb{R} \to \mathbb{R}$.

(a) If f is continuous, then the set

$$\Big\{\big(x, f(x)\big) \colon x \in \mathbb{R}\Big\}$$

(namely the **graph** of f) is closed with respect to the Euclidean metric on $\mathbb{R}^2$.

(b) The converse fails.

(c) The graph of f may fail to be closed.

Two different metrics on the same set may determine the same or different open sets.

Example 5. On $\mathbb{R}^2$, let d_1 be the Euclidean metric, and let

d_2 be the metric $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \max_i |x_i - y_i|$. Then

$$\begin{aligned} d_2(\boldsymbol{x}, \boldsymbol{y}) = \max_i \sqrt{|x_i - y_i|^2} &\leqslant \sqrt{|x_0 - y_0|^2 + |x_1 - y_1|^2} \\ &= d_1(\boldsymbol{x}, \boldsymbol{y}) \leqslant \sqrt{2} \cdot d_2(\boldsymbol{x}, \boldsymbol{y}). \end{aligned}$$

This means

$$\mathrm{B}_1(\boldsymbol{x}; r) \subseteq \mathrm{B}_2(\boldsymbol{x}; r) \subseteq \mathrm{B}_1\left(\boldsymbol{x}; \frac{r}{\sqrt{2}}\right),$$

so every point has the same neighborhoods with respect to d_1 and d_2, and therefore the same sets are open with respect to the two metrics.

Example 6. The set $\{x \in \mathbb{Q} \colon |x| < 1\}$, which in $\mathbb{Q}$ is a Euclidean neighborhood of 0, is not a p-adic neighborhood, since for every positive real number r, there is n in $\mathbb{Z}$ such that $p^n > \max(1, r^{-1})$, so $|p^n|_p = p^{-n} < r$, although $|p^n| > 1$.

Exercise 2. Show that the Euclidean and New-York metrics on $\mathbb{R}^2$ determine the same open sets.

3 The Cantor set

For every set Ω, the **power set** of Ω is given by

$$\mathscr{P}(\Omega) = \{\text{subsets of } \Omega\} = \{X \colon X \subseteq \Omega\}. \tag{9}$$

The **symmetric difference** of two subsets of Ω is given by

$$X \mathbin{\triangle} Y = (X \smallsetminus Y) \cup (Y \smallsetminus X) = (X \cup Y) \smallsetminus (X \cap Y).$$

We let ω be the set $\{0, 1, 2, \dots\}$ of **natural numbers.** It will sometimes be convenient to treat each n in ω as the set $\{x \in \omega \colon x < n\}$ of its predecessors in ω: more simply,

$$n = \{0, \dots, n-1\}. \tag{10}$$

In particular, $0 = \varnothing$.

Exercise 3. (a) An ultrametric on $\mathscr{P}(\omega)$ is given by the rule

$$d(X, Y) = \begin{cases} \dfrac{1}{2^{\min(X \triangle Y)}}, & \text{if } X \neq Y, \\ 0, & \text{if } X = Y. \end{cases}$$

This can be called the **Gromov–Hausdorff metric.**

(b) Another metric on $\mathscr{P}(\omega)$ is given by the rule

$$d^*(X, Y) = \sum_{i \in X \triangle Y} \frac{1}{2^i},$$

but this is not an ultrametric.

(c) The two metrics nonetheless have the same open sets.

By definition, a function from one metric space to another is **continuous** if, under it, the inverse image of every open set is open. By Theorem 2, this agrees with the usual definition when each space is $\mathbb{R}$.

Example 7 (The Cantor set)**.** The function f from $\mathscr{P}(\omega)$ to $\mathbb{R}$ given by

$$f(X) = \sum_{k \in X} \frac{2}{3^{k+1}}.$$

has range called the **Cantor set.** Some elements of this set are shown, with their pre-images under f, in Figure 1. For

ω	1
$\{0,1,2\}$	$2/3 + 2/3^2 + 2/3^3$
$\omega \setminus \{2\}$	$2/3 + 2/3^2 + 1/3^3$
$\{0,1\}$	$2/3 + 2/3^2$
$\omega \setminus \{1\}$	$2/3 + 1/3^2$
$\{0,2\}$	$2/3 + 2/3^3$
$\omega \setminus \{1,2\}$	$2/3 + 1/3^3$
$\{0\}$	$2/3$
X	$\sum_{k \in X} \frac{2}{3^{k+1}}$
$\omega \setminus \{0\}$	$1/3$
$\{1,2\}$	$2/3^2 + 2/3^3$
$\omega \setminus \{0,2\}$	$2/3^2 + 1/3^3$
$\{1\}$	$2/3^2$
$\omega \setminus \{0,1\}$	$1/3^2$
$\{2\}$	$2/3^3$
$\omega \setminus \{0,1,2\}$	$1/3^3$
$\varnothing$	0

Figure 1: The Cantor set

every X in $\mathscr{P}(\omega)$, for every k in ω, by (10), X is the disjoint union of $X \cap k$, which is $\{i \in X : i < k\}$, and $X \smallsetminus k$, which is $\{n \in X : k \leqslant n\}$. In particular,

$$f(X) = f(X \cap k) + f(X \smallsetminus k). \tag{11}$$

Say also $Y \in \mathscr{P}(\omega)$ and $k = \min(X \bigtriangleup Y)$ and $k \in Y$. Then

$$\begin{gathered} f(X \cap k) = f(Y \cap k), \\ f(X \smallsetminus k) \leqslant f\big(\omega \smallsetminus (k+1)\big), \\ f\big(\{k\}\big) \leqslant f(Y \smallsetminus k). \end{gathered}$$

Since

$$f\big(\omega \smallsetminus (k+1)\big) = \sum_{k+1 \leqslant n} \frac{2}{3^{n+1}} = \frac{1}{3^{k+1}} < \frac{2}{3^{k+1}} = f\big(\{k\}\big), \tag{12}$$

it follows that, for distinct subsets X and Y of ω,

$$f(X) < f(Y) \iff \min(X \bigtriangleup Y) \in Y. \tag{13}$$

In particular, f is injective. Now give $\mathscr{P}(\omega)$ the Gromov–Hausdorff metric from Exercise 3, and call this metric d. We shall show that f is continuous with respect to d and the Euclidean metric on $\mathbb{R}$. Given A in $\mathscr{P}(\omega)$ and positive r in $\mathbb{R}$, shall find positive s and t in $\mathbb{R}$ such that

$$\begin{gathered} \forall X\ \big(X \in \mathrm{B}(A; s) \Rightarrow f(X) < f(A) + r\big), \\ \forall X\ \big(X \in \mathrm{B}(A; t) \Rightarrow f(X) > f(A) - r\big); \end{gathered}$$

and then

$$\forall X\ \Big(X \in \mathrm{B}\big(A; \min(s,t)\big) \Rightarrow f(X) \in \mathrm{B}\big(f(A); r\big)\Big),$$

so f will be continuous at A. To find s and t, we consider two cases each.

1. Suppose first there is B in $\mathscr{P}(\omega)$ such that

$$f(A) < f(B) \leqslant f(A) + r. \tag{14}$$

Then $\min(A \vartriangle B) \in B$ by (13). If $d(A, C) < d(A, B)$, then, since all non-equilateral triangles are isosceles, with third side shorter than the two equal sides, we must have $d(B, C) = d(A, B)$, so that $\min(B \vartriangle C) \in B$, and therefore $f(C) < f(B)$. Thus we may let $s = d(A, B)$.

2. Now suppose there is no such B as in (14). Then A must be such that, for some n in ω, $\{n, n+1, n+2, \dots\} \subseteq A$. In this case, if $d(A, C) < 1/2^{n-1}$, then either $C = A$ or $\min(A \vartriangle C) \geqslant n$, and in the latter case $\min(A \vartriangle C) \in A$, so in either case $f(C) \leqslant f(A)$. Thus we may let $s = 1/2^{n-1}$.

3. Turning to t, we suppose first there is B in $\mathscr{P}(\omega)$ such that

$$f(A) - r \leqslant f(B) < f(A). \tag{15}$$

Then $\min(A \vartriangle B) \notin B$. If $d(A, C) < d(A, B)$, then $d(B, C) = d(A, B)$, so that $\min(B \vartriangle C) = \min(A, B)$; since this is not in B, we must have $f(B) < f(C)$. Thus we may let $t = d(A, B)$.

4. Now suppose there is no such B as in (15). Then A must be such that, for some n in ω, $\{n, n+1, n+2, \dots\} \cap A = \varnothing$. In this case, if $d(A, C) < 1/2^{n-1}$, then either $C = A$ or $\min(A \vartriangle C) \geqslant n$, and in the latter case $\min(A \vartriangle C) \in C$, so in either case $f(A) \leqslant f(C)$. Thus we may let $t = 1/2^{n-1}$.

Having found s and t as desired, we can conclude that f is continuous. Since f is injective, f^{-1} is well-defined as a function on the Cantor set. This function too is continuous. Indeed, let $A \in \mathscr{P}(\omega)$ and $n \in \omega$. If

$$f(A \cap n) \leqslant f(X) \leqslant f(A \cap n) + f(\omega \smallsetminus n),$$

then

$$\min(A \mathbin{\triangle} X) \notin A \cap n,$$
$$\min(A \mathbin{\triangle} X) \in (A \cap n) \cup (\omega \smallsetminus n),$$

and so

$$\min(A \mathbin{\triangle} X) > n, \qquad\qquad d(A, X) \leqslant \frac{1}{2^n}.$$

Thus if we define

$$I = \bigl[f(A \cap n), f(A \cap n) + f(\omega \smallsetminus n)\bigr] = [a, b],$$

then

$$\begin{gathered} f(X) \in I \implies d(A, X) \leqslant \frac{1}{2^n}, \\ f(A) \in I. \end{gathered} \tag{16}$$

The closed interval I, namely $[a, b]$, contains $f(A)$. We show how to replace I with an open interval that includes it. If

$$f(X) < a = f(A \cap n),$$

then for some m in ω,

$$\min(A \mathbin{\triangle} X) = m < n, \qquad\qquad m \in A \smallsetminus X,$$

and so

$$f(X) \leqslant f(A \cap m) + f\bigl(\omega \smallsetminus (m+1)\bigr) = f(A \cap m) + \frac{1}{3^{m+1}},$$
$$f(X) + \frac{1}{3^n} \leqslant f(X) + \frac{1}{3^{m+1}} \leqslant f\bigl(A \cap (m+1)\bigr) \leqslant f(A \cap n).$$

Thus

$$a - \frac{1}{3^n} < f(X) \implies a \leqslant f(X). \tag{17}$$

Finally, if

$$f(X) > b = f(A \cap n) + f(\omega \smallsetminus n),$$

then again for some m in ω,

$$\min(A \mathbin{\triangle} X) = m < n, \qquad\qquad m \in X \smallsetminus A,$$

and so

$$\begin{aligned}
f(X) &\geqslant f(A \cap m) + \frac{2}{3^{m+1}} \\
&= f(A \cap m) + f\big(\omega \smallsetminus (m+1)\big) + \frac{1}{3^{m+1}} \\
&= f(A \cap m) + f\big(n \smallsetminus (m+1)\big) + f\big(\omega \smallsetminus n\big) + \frac{1}{3^{m+1}} \\
&\geqslant f(A \cap n) + f(\omega \smallsetminus n) + \frac{1}{3^n}.
\end{aligned}$$

Thus

$$f(X) < b + \frac{1}{3^n} \implies f(X) \leqslant b.$$

Combining with (17) yields

$$a - \frac{1}{3^n} < f(X) < b + \frac{1}{3^n} \implies f(X) \in I.$$

This with (16) shows that f^{-1} is continuous at $f(A)$.

4 Topological spaces

We now define a **topology** on a set as a collection of subsets of the set that contains

1) the union of every collection (including the empty collection) of its members and

2) the intersection of every finite collection (including the empty collection) of its members.

We can rewrite the second condition as being that the collection contains:

a) the intersection of any two of its members, and
b) the whole set.

If τ is a topology on a set Ω, we call the pair (Ω, τ) a **topological space,** and the elements of τ are the **open subsets** of the space.[5] If τ can be understood, we may refer to Ω itself as a topological space. Note also that $\bigcup \tau = \Omega$, so Ω can be recovered from τ.

For a more symbolic presentation, having in mind (9), we define

$$\mathscr{P}_\omega(\Omega) = \{\text{finite subsets of } \Omega\} = \{X \colon X \subseteq \Omega \;\;\&\;\; |X| < \infty\}.$$

Here ω, the set of natural numbers, can be understood as the smallest infinite *cardinal number* (see §11, page 48). Thus $\mathscr{P}_\omega(\Omega)$ comprises the subsets of Ω that are smaller than ω. By definition then, τ is a topology on Ω if $\tau \subseteq \mathscr{P}(\Omega)$ and

$$\forall \mathscr{X} \;\left(\mathscr{X} \in \mathscr{P}(\tau) \Rightarrow \bigcup \mathscr{X} \in \tau\right),$$
$$\forall \mathscr{X} \;\left(\mathscr{X} \in \mathscr{P}_\omega(\tau) \Rightarrow \bigcap \mathscr{X} \in \tau\right).$$

[5]It is an historical peculiarity that many writers use X as a name for a specific topological space. This makes X a constant, technically, when the lower-case x has been used since the time of Descartes as a variable. I try to use lower-case letters for individuals (numbers, points, elements); upper-case letters for sets of these; and fancy letters for collections of sets. The letters x and y are variables in every case: so X and Y, $\mathscr{X}$ and $\mathscr{Y}$ are variables. There seem to be no standard variables for functions, so f and g are used in (8).

In any topological space, a **neighborhood** of a point is a subset of the space that includes an open set that contains the point.

Theorem 4. *In any topological space, a set is open if and only if it is a neighborhood of each of its points.*

Proof. Since sets include themselves, every open set is immediately a neighborhood of each of its points. Suppose a set E is a neighborhood of each of its points. Then for each point a in E, there is an open set O_a such that $a \in O_a$ and $O_a \subseteq E$. Then

$$E = \bigcup_{x \in E} O_x,$$

so, being a union of open sets, E is open. □

In the proof, there may be infinitely many points a in E, and for each a, there may be more than one open set O such that $a \in O$ and $O \subseteq E$. We have to choose one such O for each a. One formulation of the **Axiom of Choice** is precisely that we can do this.

Example 8. Every set with at least two elements can be given two different topologies:

1) the **discrete topology,** in which every set is open;
2) the **trivial topology,** in which only the empty set and the whole space are open.

Example 9. Every infinite set has a topology that is neither discrete nor trivial: the **cofinite topology,** in which the empty set and the complement of every finite set are open.

5 Closures

In every topological space, the complement of an open set is called **closed.** Thus

1) the intersection of every collection of closed sets is closed,
2) the union of every finite collection of closed sets is closed.

The **closure** of a set is the smallest closed set that includes it. This definition makes sense, precisely because the intersection of every collection of closed sets is closed, and every member of the collection includes the intersection. In a topological space (Ω, τ), we may write

$$X^{\mathrm{c}} = \Omega \smallsetminus X = \{x \in \Omega \colon x \notin X\}.$$

Then, for the closure of a set E, we write

$$\overline{E} = \bigcap\{X \colon X^{\mathrm{c}} \in \tau \;\;\&\;\; E \subseteq X\}.$$

Theorem 5. *In any topological space Ω, closure is an operation φ on $\mathscr{P}(\Omega)$ satisfying*

1) $X \subseteq \varphi(X)$,
2) $\varphi(\varphi(X)) = \varphi(X)$,
3) $\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y)$,
4) $\varphi(\varnothing) = \varnothing$.

Conversely, if some operation φ on $\mathscr{P}(\Omega)$ has these properties, then φ is the closure operation for a topology on Ω.

Example 10. The operation

$$X \mapsto \begin{cases} X, & \text{if } |X| < \infty, \\ \Omega, & \text{if } |X| = \infty, \end{cases}$$

satisfies the conditions of the theorem and is therefore the closure operation for a topology on Ω. This topology is the cofinite topology.

Exercise 4. In a metric space (M, d),

(a) $\overline{X} = \bigl\{y \in M \colon \inf\{d(y, z) \colon z \in X\} = 0\bigr\}$,

(b) $\overline{\mathrm{B}(a; r)} \subseteq \{x \in M \colon d(a, x) \leqslant r\}$, but

(c) the last inclusion can be strict.

Example 11. Always

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B};$$

but in $\mathbb{R}$ with the usual Euclidean topology,

$$\overline{\mathbb{Q}} = \mathbb{R}, \qquad \overline{\mathbb{Q}^{\mathrm{c}}} = \mathbb{R}, \qquad \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^{\mathrm{c}}} = \mathbb{R}, \qquad \overline{\mathbb{Q} \cap \mathbb{Q}^{\mathrm{c}}} = \varnothing.$$

Exercise 5. Let $B \subseteq \Omega$.

(a) Show that the function

$$X \mapsto \begin{cases} X \cup B, & \text{if } X \neq \varnothing, \\ \varnothing, & \text{if } X = \varnothing, \end{cases}$$

is the closure operation for a topology on Ω.

(b) What are the open sets in this topology?

The **interior** of a set is the largest open set that it includes. The definition makes sense, like the definition of the closure of a set. In a topological space (Ω, τ), we may write

$$E^{\circ} = \bigcap\{X \colon X \in \tau \ \ \& \ \ X \subseteq E\}.$$

Theorem 6. *The interior of a set consists precisely of the points of which the set is a neighborhood.*

Proof. For all subsets E of a topological space, for all points a of the space, the following statements are equivalent:

1. E is a neighborhood of a.

2. $a \in O$ and $O \subseteq E$, for some open set O.
3. $a \in E^\circ$. □

Theorem 7. *For all subsets E of a topological space,*

$$(E^\circ)^c = \overline{E^c}, \tag{18}$$
$$E^\circ = \overline{E^c}^c, \tag{19}$$
$$(E^c)^\circ = \overline{E}^c. \tag{20}$$

Proof. If the topology is τ, we compute

$$\begin{aligned}(E^\circ)^c &= \left(\bigcup\{X \in \tau \colon X \subseteq E\}\right)^c \\ &= \bigcap\{X^c \colon X \in \tau \ \& \ X \subseteq E\} \\ &= \bigcap\{X^c \colon X \in \tau \ \& \ E^c \subseteq X^c\} \\ &= \bigcap\{X \colon X^c \in \tau \ \& \ E^c \subseteq X\} = \overline{E^c},\end{aligned}$$

so we have (18) and then (19). Replacing E with E^c yields (20). □

Exercise 6. (a) Show that

$$\overline{X^\circ} = \overline{\overline{X^\circ}^\circ}. \tag{21}$$

(b) Conclude that

$$\overline{\overline{X}^c} = \overline{\overline{\overline{\overline{X}^c}^c}^c}, \tag{22}$$

and therefore the smallest collection of subsets of a topological space that contains a given set and is closed under the operations of taking closures, complements, and interiors has no more than 14 members.

(c) If the space is $\mathbb{R}$, there is an example with exactly 14 members.

A **limit point** of a subset of a topological space is a point whose every neighborhood contains an element of the subset other than the point itself. The set of all limit points of a set A can be denoted by

$$A';$$

this is the **Cantor–Bendixson derivative** of A.

Theorem 8. $\overline{A} = A \cup A'$.

Proof. By Theorem 7 we have

$$\begin{aligned} x \in \overline{A} &\iff x \notin (A^{\mathrm{c}})^{\circ} \\ &\iff A^{\mathrm{c}} \text{ is not a neighborhood of } x \\ &\iff \text{ every neighborhood of } x \text{ contains a point of } A \\ &\iff x \in A \text{ or } x \in A' \\ &\iff x \in A \cup A'. \end{aligned}$$

□

Example 12. In $\mathbb{R}$, we have

$$\mathbb{Q}' = \mathbb{R}, \qquad \left\{\frac{1}{n} \colon n \in \mathbb{N}\right\} = \{0\}, \qquad \{0\}' = \varnothing;$$

in $\mathbb{R}^2$, if $A = \{(1/m, 1/n) \colon (m, n) \in \mathbb{N}^2\}$, then

$$A' = \big\{(0,0)\big\} \cup \left\{\left(\frac{1}{m}, 0\right) \colon m \in \mathbb{N}\right\} \cup \left\{\left(0, \frac{1}{n}\right) \colon n \in \mathbb{N}\right\},$$
$$A'' = \big\{(0,0)\big\}, \qquad A''' = \varnothing.$$

Example 13 (The Zariski topology)**.** Letting

$$\Omega = \{0\} \cup \{p \in \mathbb{N} \colon p \text{ is prime}\}, \tag{23}$$

we are going to define a topology on Ω so that

$$\overline{\{0\}} = \Omega, \qquad \overline{\{p\}} = \{p\}, \qquad \Omega' = \Omega. \tag{24}$$

The topology arises naturally as follows. If a and b are in $\mathbb{Z}$, we define

$$\begin{aligned}
(a) &= \{ax \colon x \in \mathbb{Z}\},\\
(a,b) &= \{ax + by \colon (x,y) \in \mathbb{Z}^2\},\\
\gcd(a,b) &= \begin{cases} \min\bigl(\mathbb{N} \cap (a,b)\bigr), & \text{if one of } a \text{ and } b \text{ is not } 0,\\ 0, & \text{if both } a \text{ and } b \text{ are } 0. \end{cases}
\end{aligned}$$

Consequently

$$(a,b) = \bigl(\gcd(a,b)\bigr).$$

The subsets (a) and (a,b) are **ideals** of $\mathbb{Z}$, because they are additive subgroups of $\mathbb{Z}$ that are closed under multiplication by arbitrary elements of $\mathbb{Z}$. A *proper* ideal I of $\mathbb{Z}$ is called **prime** if

$$xy \in I \land x \notin I \implies y \in I.$$

If p is a prime number, then (p) is a prime ideal, by Euclid's Lemma, whereby

$$p \mid ab \land p \nmid a \implies p \mid b.$$

Although 0 is not a prime number, (0) is a prime ideal because

$$ab = 0 \land a \neq 0 \implies b = 0.$$

There are no other prime ideals of $\mathbb{Z}$. In particular, since (1) is an improper ideal, it is not prime. We denote the set of prime ideals of $\mathbb{Z}$ by

$$\operatorname{Spec}(\mathbb{Z});$$

this is the **spectrum** of $\mathbb{Z}$, and we shall topologize it by defining

$$\begin{aligned}
\mathrm{V}(a) &= \{I \in \operatorname{Spec}(\mathbb{Z})\colon a \in I\} \\
&= \begin{cases} \bigl\{(p)\colon p \mid a\bigr\}, & \text{if } a \neq 0, \\ \operatorname{Spec}(\mathbb{Z}), & \text{if } a = 0. \end{cases}
\end{aligned}$$

In particular, $\mathrm{V}(a)$ is finite when $a \neq 0$. Moreover, for any *nonempty* set P of prime numbers,

$$\bigl\{(p)\colon p \in P\bigr\} = \mathrm{V}\left(\prod P\right),$$

where $\prod P$ is the product of the elements of P. By definition, $\prod \varnothing = 1$, and $\mathrm{V}(1) = \varnothing$. Then the sets $\mathrm{V}(a)$ are the closed sets of a topology on $\operatorname{Spec}(\mathbb{Z})$. A way to confirm this is to note first

$$\begin{gathered}
\varnothing = \mathrm{V}(1), \\
\mathrm{V}(a) \cup \mathrm{V}(b) = \mathrm{V}(ab).
\end{gathered}$$

We can define $\gcd(a, b, c)$ and so forth in the obvious way, and then, for every subset A of $\mathbb{Z}$, possibly infinite, we can define

$$\gcd(A) = \begin{cases} \min\bigl\{\gcd(X)\colon X \in \mathscr{P}_\omega(A) \smallsetminus \{\varnothing\}\bigr\}, & \text{if } A \neq \varnothing, \\ 0 & \text{if } A = \varnothing. \end{cases}$$

Then

$$\bigcap_{x \in A} \mathrm{V}(x) = \mathrm{V}\bigl(\gcd(A)\bigr),$$

even when A is empty. Consequently the sets $\mathrm{V}(a)$ are the closed sets of a topology on $\mathrm{Spec}(\mathbb{Z})$, called the **Zariski topology.** Such a topology is defined on the spectrum of prime ideals of any commutative ring. When the ring is $\mathbb{Z}$, the Zariski topology is like the cofinite topology, except that (0) belongs to every nonempty open set. Consequently, if we confuse Ω as in (23) with $\mathrm{Spec}(\mathbb{Z})$ in the obvious way, we have (24).

6 Bases

A topology τ_0 on a set is **coarser** or **weaker** than a topology τ_1 on the same set, and τ_1 is **finer** or **stronger** than τ_0, if τ_1 includes τ_0.

Example 14 (The Tarski topology)**.** If again Ω is the subset of $\mathbb{Z}$ given in (23), we shall define a topology on Ω so that

$$\overline{\{0\}} = \{0\}, \qquad \overline{\{p\}} = \{p\}, \qquad \Omega' = \{0\}. \tag{25}$$

This will be the *coarsest* topology in which all of the sets $\{p\}$ and their complements are open. By the next theorem, such a topology exists; but without this, we can just describe the topology. Since

$$\{0\} = \bigcap_{p \text{ prime}} \{p\}^{\mathrm{c}},$$

this set must be closed. Since each set $\{p\}$ is closed, all finite subsets of Ω must be closed, and therefore all cofinite subsets of Ω must be open. Since each set $\{p\}$ is open, all subsets of $\Omega \setminus \{0\}$ must be open. The open sets that we have found do in fact constitute a topology, and in this topology, $\{0\}$ is not open. Thus we have (25).

The topology on Ω that we have found can be understood to arise as follows. Let σ_p be the equation $1+\cdots+1=0$, where 1 occurs p-many times; and then let $\neg\sigma_p$ be the corresponding inequation $1+\cdots+1\neq 0$. We shall refer to the σ_p and $\neg\sigma_p$ as **sentences.** Every field K has a **theory,** $\mathrm{Th}(K)$, consisting of the sentences that are true in K. Then for every prime number ℓ,

$$\begin{gathered}\mathrm{Th}(\mathbb{F}_\ell)=\{\sigma_\ell\}\cup\{\neg\sigma_p\colon p\neq\ell\},\\ \mathrm{Th}(\mathbb{Q})=\{\neg\sigma_p\colon p\text{ is prime}\}.\end{gathered}$$

The theory of every field is one of these theories. Let S be the set of all theories of fields; then we have a bijection from Ω to S that takes p to $\mathrm{Th}(\mathbb{F}_p)$ and 0 to $\mathrm{Th}(\mathbb{Q})$. The inverse takes $\mathrm{Th}(K)$ to the **characteristic** of K, or $\mathrm{char}(K)$. We can use the bijection to carry over to S the topology on Ω that we found above. The topology so obtained on S, the **Tarski topology,** is the coarsest in which the sets $\{T\in S\colon\sigma\in T\}$ are closed, as σ ranges over the sentences.

Theorem 9. *For any set Ω, for any subset $\mathscr{S}$ of $\mathscr{P}(\Omega)$, if we let*

$$\mathscr{B}=\Bigl\{\bigcap Y : Y\in\mathscr{P}_\omega(S)\Bigr\},$$

then the set

$$\Bigl\{\bigcup\mathscr{X} : \mathscr{X}\in\mathscr{P}(\mathscr{B})\Bigr\}$$

is a topology on Ω and is the coarsest topology on Ω that includes $\mathscr{S}$.

Proof. Let the set in question be called $\mathscr{T}$. If τ is a topology on Ω that includes $\mathscr{S}$, then $\mathscr{B}\subseteq\tau$, and then τ must also include $\mathscr{T}$. We show now that $\mathscr{T}$ is a topology on Ω. Suppose $\mathscr{A}$ and $\mathscr{C}$ are subsets of $\mathscr{B}$. Then

$$\bigcup\mathscr{A}\cap\bigcup\mathscr{C}=\bigcup\{X\cap Y\colon X\in\mathscr{A}\ \ \&\ \ Y\in\mathscr{C}\},$$

which is in $\mathscr{T}$. Since also Ω is in $\mathscr{B}$ and is therefore in $\mathscr{T}$, this is closed under finite intersections. Finally, suppose $\boldsymbol{A}$ is a collection of subsets of $\mathscr{B}$. Then $\bigcup\boldsymbol{A}$ is a subset of $\mathscr{B}$, and

$$\begin{aligned}
&\bigcup\left\{\bigcup\mathscr{X}:\mathscr{X}\in\boldsymbol{A}\right\}\\
&=\left\{z\colon\exists\mathscr{X}\ \left(\mathscr{X}\in\boldsymbol{A}\ \ \&\ \ z\in\bigcup\mathscr{X}\right)\right\}\\
&=\{z\colon\exists\mathscr{X}\ \exists Y\ \ (\mathscr{X}\in\boldsymbol{A}\ \ \&\ \ Y\in\mathscr{X}\ \ \&\ \ z\in Y)\}\\
&=\{z\colon\exists Y\ \exists\mathscr{X}\ \ (\mathscr{X}\in\boldsymbol{A}\ \ \&\ \ Y\in\mathscr{X}\ \ \&\ \ z\in Y)\}\\
&=\left\{z\colon\exists Y\ \left(Y\in\bigcup\boldsymbol{A}\ \ \&\ \ z\in Y\right)\right\}\\
&=\bigcup\bigcup\boldsymbol{A},
\end{aligned}$$

which is therefore in $\mathscr{T}$. Thus $\mathscr{T}$ is closed under arbitrary unions. □

In the theorem, $\mathscr{B}$ is a **base** or **basis** for the topology $\mathscr{T}$, because every member of $\mathscr{T}$ is a union of members of $\mathscr{B}$. Also $\mathscr{S}$ is a **sub-base sub-basis** for $\mathscr{T}$. By the theorem, every set is a sub-base for some topology.

Theorem 10. *For any set Ω, a subset $\mathscr{B}$ of $\mathscr{P}(\Omega)$ is a base for a topology on Ω if and only if $\bigcup\mathscr{B}=\Omega$ and, for any two elements A and C of $\mathscr{B}$, for any d in $A\cap C$, for some E in $\mathscr{B}$,*

$$E\subseteq A\cap C,\qquad\qquad d\in E.$$

Exercise 7. Show that the collection of open squares

$$(a, a+\delta) \times (b, b+\delta)$$

is a base for a topology on $\mathbb{R}^2$.

Theorem 11. *For any topological space (Ω, τ), for any subset A of Ω, the set*

$$\{X \cap A \colon X \subseteq \Omega\}$$

is a topology on A. A subset X of A is closed in this topology if and only if, for some closed subset F of Ω,

$$X = A \cap F.$$

In any case, if $\overline{X}$ is the closure of X in Ω, then $A \cap \overline{X}$ is its closure in A.

In the theorem, A with its topology is a **subspace** of (Ω, τ).

As noted earlier, a function f from a topological space A to a topological space C is **continuous** if $f^{-1}[U]$ is open in A for every open subset U of C, that is, the inverse image of every open set is open. Since

$$f^{-1}[X]^{\mathrm{c}} = f^{-1}[X^{\mathrm{c}}],$$

the function f is continuous if and only if the inverse image of every closed set is closed. If B is a subspace of C and $f[A] \subseteq B$, then f is still continuous as a function from A to B. If $f[A] = B$, and f is injective, and f^{-1} is continuous, then f is called a **homeomorphism** from A to B, and the spaces A and B are **homeomorphic** to one another. In Example 7, we showed that $\mathscr{P}(\omega)$, with the topology induced by the Gromov–Hausdorff metric, is homeomorphic to the Cantor set with the Euclidean topology induced from $\mathbb{R}$. In Examples 13 and 14, a set Ω was given two different topologies, making it respectively homeomorphic to two different spaces, having the Zariski and Tarski topologies respectively.

Example 15. All open intervals of $\mathbb{R}$ are homeomorphic to one another.

7 Products

If (A, τ_0) and (B, τ_1) are two topological spaces, then $\tau_0 \times \tau_1$ is a basis for a topology, called the **product topology,** on $A \times B$. The basis thus consists of the products of open subsets of A and B.

Exercise 8. The product topology on $\mathbb{R} \times \mathbb{R}$ is the Euclidean topology on $\mathbb{R}^2$.

We shall consider the product of an arbitrary, possibly infinite number of spaces. If we are given an indexed family $(\Omega_i : i \in I)$ of sets, we define its **product** by the identity

$$\prod_{i \in I} \Omega_i = \big\{(x_i : i \in I) : \forall i \ (i \in I \Rightarrow x_i \in \Omega_i)\big\}.$$

Here $(x_i : i \in I)$ is just the function $i \mapsto x_i$ having domain I. The range of the function is a subset of $\bigcup_{i \in I} \Omega_i$.

Suppose now each Ω_i has the topology τ_i. As before, the product of the τ_i is a basis of a topology on the product of the Ω_i; but this topology is not generally the one that we shall be interested in.

For each j in I, there is a **projection** π_j from $\prod_{i \in I} \Omega_i$ onto Ω_j given by

$$\pi_j(x_i : i \in I) = x_j.$$

The **product topology** or **Tychonoff topology** on $\prod_{i\in I}\Omega_i$ is the coarsest topology in which the projections are continuous. This means the topology has a sub-base

$$\left\{{\pi_j}^{-1}[U_j]\colon j\in I\ \ \&\ \ U_j\in\tau_j\right\}.$$

Then the topology has a base

$$\left\{\bigcap_{j\in J}{\pi_j}^{-1}[U_j]\colon J\in\mathscr{P}_\omega(I)\ \ \&\ \ U_j\in\tau_j\right\}.$$

We also have

$$\bigcap_{j\in J}{\pi_j}^{-1}[U_j]=\prod_{i\in I}\Omega_i^*,\ \text{ where } \Omega_i^*=\begin{cases}U_i, & \text{if } i\in J,\\ \Omega_i, & \text{if } i\in I\smallsetminus J.\end{cases}$$

In case all of the spaces Ω_i are the same space Ω, we let

$$\prod_{i\in I}\Omega_i=\Omega^I;$$

this is the space of functions from I to Ω.

Example 16. The simplest nontrivial example of an infinite product of spaces is 2^ω, where $2=\{0,1\}$. Here 2 is given the discrete topology (Example 8, page 19), so that both $\{0\}$ and $\{1\}$ are open. Then the Tychonoff topology on 2^ω has a sub-basis consisting of, for each n in ω, the sets

$$\{f\in 2^\omega\colon f(n)=0\},\qquad\qquad \{f\in 2^\omega\colon f(n)=1\}.$$

We can write these sets also as

$$\begin{gathered}\{(x_0,\dots,x_{n-1},0,x_{n+1},\dots)\colon x_i\in 2\},\\ \{(x_0,\dots,x_{n-1},1,x_{n+1},\dots)\colon x_i\in 2\}.\end{gathered}$$

Since each of these is the complement of the other, the sets in the sub-base are closed as well as open; in a word, they are **clopen.** In the general notation for projections of products, the sub-basic sets are of the forms

$$\pi_n{}^{-1}\big[\{0\}\big], \qquad \pi_n{}^{-1}\big[\{1\}\big],$$

which we can write more simply as

$$\pi_n{}^{-1}(0), \qquad \pi_n{}^{-1}(1).$$

For further discussion, we shall again understand each n in ω as in (10). The Tychonoff topology on 2^ω has a base consisting of every finite intersection of sets in the sub-base above. Such an intersection has the form

$$\bigcap_{k<m} \pi_{i_k}{}^{-1}(e_k), \tag{26}$$

where $m \in \omega$, $(e_k : k < m) \in 2^m$, $(i_k : k < m) \in \omega^m$, and

$$i_0 < i_1 < \cdots < i_{m-1}.$$

An element of the intersection (26) might be written, somewhat vaguely, as

$$(\dots, e_0, \dots, e_1, \dots, e_{m-1}, \dots).$$

A special case of (26) is

$$\bigcap_{k<n} \pi_k{}^{-1}(e_k), \tag{27}$$

where $n \in \omega$ and $(e_k : k < n) \in 2^n$; we might write the special case (27) also as either of

$$\{(e_0, \dots, e_{n-1}, x_0, x_1, \dots) : (x_i : i \in \omega) \in 2^\omega\},$$
$$\left\{f \in 2^\omega : \bigwedge_{i<n} f(i) = e_i\right\}.$$

The intersection (26) is a union of intersections as in (27). The latter then compose a basis for the Tychonoff topology on 2^ω.

Example 17. There is a bijection

$$f \mapsto \{i \in \omega \colon f(i) = 1\} \tag{28}$$

from 2^ω to $\mathscr{P}(\omega)$. The inverse is $Y \mapsto \chi_Y$, where, if $A \subseteq \omega$,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in \omega \smallsetminus A. \end{cases}$$

We can define the **Tychonoff topology** on $\mathscr{P}(\omega)$ to be that topology with respect to which the bijection $Y \mapsto \chi_Y$ from $\mathscr{P}(\omega)$ to 2^ω is a homeomorphism. Then the Tychonoff topology on $\mathscr{P}(\omega)$ has a basis consisting of the sets

$$\left\{X \subseteq \omega \colon \bigwedge_{i<n} (i \in X \Leftrightarrow e_i = 1)\right\}, \tag{29}$$

where $n \in \omega$ and $(e_k \colon k < n) \in 2^n$. If A is an element of the set in (29), then that set is $\{X \subseteq \omega \colon X \cap n = A \cap n\}$.

Exercise 9. In the Tychonoff topology on $\mathscr{P}(\omega)$, are the following sets open, closed, both, or neither?

(a) $\{X \subseteq \omega \colon 16 \in X \land 17 \notin X\}$
(b) $\{X \subseteq \omega \colon \forall y\ (y \in \omega \Rightarrow 2y \in X)\}$
(c) $\{X \subseteq \omega \colon \forall y\ (y \in X \Rightarrow y + 1 \in X)\}$

Exercise 10. Show that the Tychonoff topology on $\mathscr{P}(\omega)$ is precisely the topology induced by the Gromov–Hausdorff metric.

Theorem 12 (Cantor Intersection Theorem)**.** *Every decreasing sequence of nonempty closed bounded subsets of* $\mathbb{R}$ *has nonempty intersection. That is, if*

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots ,$$

where each F_n *is a nonempty closed subset of* $\mathbb{R}$*, and* F_0 *is bounded, then*

$$\bigcap_{n\in\omega} F_n \neq \varnothing.$$

Proof. Let

$$a_n = \inf F_n.$$

Then the sequence $(a_n \colon n \in \omega)$ is increasing and bounded, so it has a limit b (namely $\sup_{n\in\omega} a_n$). Since the F_n are closed, $a_n \in F_n$. Then for all k in ω, $a_{n+k} \in F_n$. If the set $\{a_n \colon n \in \omega\}$ is finite, then b belongs to it, and therefore

$$b \in \bigcap_{n\in\omega} F_n.$$

If $\{a_n \colon n \in \omega\}$ is infinite, then b is a limit point of it, and so b is a limit point of each F_n. Therefore, by Theorem 8, b belongs to each F_n, since this is closed. □

The explanation for why the theorem is correct is that closed bounded subsets of $\mathbb{R}$ are *compact* (see §12, page 55). We shall see similarly that, in the Tychonoff topology, $\mathscr{P}(\omega)$ is compact.

Example 18. The Cantor Intersection Theorem may fail if the F_n are not both closed and bounded, since

$$\bigcap_{n\in\omega} [n, \infty) = \varnothing, \qquad \bigcap_{n\in\omega} \left(0, \frac{1}{n+1}\right) = \varnothing.$$

See Exercise 16 (page 57).

Example 19 (Spaces of groups)**.** If G is a group, and $\mathscr{P}(G)$ has the Tychonoff topology, then the subset $\{H \in \mathscr{P}(G) \colon H \leqslant G\}$ consisting of subgroups of G is closed, since the subset is the intersection

$$\{X \in \mathscr{P}(G) \colon \mathrm{e} \in X\} \cap \bigcap_{(a,b)\in G^2} \{X \in \mathscr{P}(G) \colon a \in X \wedge b \in X \Rightarrow ab^{-1} \in X\},$$

and each set $\{X \colon a \in X \wedge b \in X \Rightarrow ab^{-1} \in X\}$ is the union

$$\{X \colon a \notin X\} \cup \{X \colon b \notin X\} \cup \{X \colon ab^{-1} \in X\}.$$

We shall see that therefore $\{H \in \mathscr{P}(G) \colon H \leqslant G\}$ is compact. Similarly the set of normal subgroups of G is closed in $\mathscr{P}(G)$ and is therefore compact. If G is a finitely generated free group, then a quotient G/N of G is a *limit group* if, in the Tychonoff topology, N is the limit of a sequence $(N_k \colon k \in \omega)$ of normal subgroups of G such that each quotient G/N_k is free. See page 47.

Example 20 (Propositional logic)**.** The Tychonoff topology on $\mathscr{P}(\omega)$ arises in propositional logic. Here we start with a set $\{P_n \colon n \in \omega\}$ of **propositional variables.** We define **propositional formulas** recursively: the propositional variables are propositional formulas, and if F and G are propositional formulas, then so are the **negation** $\neg F$ and the **conjunction** $(F \wedge G)$. Because of the use of parentheses, every propositional formula is constructed in a unique way: a negation is never also a conjunction, and if $(F \wedge G)$ and $(H \wedge K)$ are the same conjunction, then F and H must be the same formula, and likewise G and K. Therefore every propositional formula

F determines a function $X \mapsto F^X$ from $\mathscr{P}(\omega)$ to 2, according to the following recursive definition:

$$\left.\begin{aligned} {P_n}^X &= \chi_X(n) = \begin{cases} 1, & \text{if } n \in X, \\ 0, & \text{if } n \notin X, \end{cases} \\ \neg F^X &= F^X + 1, \\ (F \wedge G)^X &= F^X \cdot G^X. \end{aligned}\right\} \tag{30}$$

The operations on the right in (30) are as in the ring $\mathbb{Z}_2$, which is the field $\mathbb{F}_2$. If $F^X = 1$, we say that F is **true** in X, or X is a **model** of F. We define

$$\operatorname{Mod}(F) = \{X \in \mathscr{P}(\omega) \colon F^X = 1\};$$

its elements are just the models of F. Then

$$\begin{aligned} \operatorname{Mod}(P_n) &= \{X \subseteq \omega \colon n \in X\}, \\ \operatorname{Mod}(\neg F) &= \mathscr{P}(\omega) \smallsetminus \operatorname{Mod}(F), \\ \operatorname{Mod}(F \wedge G) &= \operatorname{Mod}(F) \cap \operatorname{Mod}(G). \end{aligned}$$

This shows that the sets of models of formulas compose a basis for the Tychonoff topology on $\mathscr{P}(\omega)$. If $\mathscr{F}$ is a collection of propositional formulas, we define

$$\operatorname{Mod}(\mathscr{F}) = \bigcap_{F \in \mathscr{F}} \operatorname{Mod}(F);$$

its elements are the **models** of $\mathscr{F}$. These sets $\operatorname{Mod}(\mathscr{F})$ are precisely the closed subsets of $\mathscr{P}(\omega)$ in the Tychonoff topology. Suppose $\mathscr{F} = \{F_n \colon n \in \omega\}$, and let

$$A_n = \operatorname{Mod}(\{F_k \colon k < n\}).$$

Then $(A_n : n \in \omega)$ is a decreasing sequence of closed subsets of $\mathscr{P}(\omega)$. If each A_n is nonempty, we shall show that their intersection is nonempty. This means $\mathscr{F}$ has a model, provided that every finite subset of $\mathscr{F}$ has a model.

8 Quotients

Example 21 (The Tarski topology)**.** As defined in Example 14 (page 26), the Tarski topology on the set of theories of fields can be understood to arise as follows. We start with the class $\boldsymbol{C}$ of all fields. This is actually a proper class, not a set. All this means is that it behaves like a set, except that it cannot be an element of any set (or class). For example, the class $\{x : x \notin x\}$ is not a set, since if it were, it would have both to be and not to be a member of itself. This will not be a problem for us. We now refer to the sentences σ_p and $\neg\sigma_p$ that we defined before as **atomic sentences.** We obtain from these all **sentences,** in a broader sense than before, just as we obtained propositional formulas from propositional variables in Example 20. Thus every atomic sentence is a sentence, and if σ and ρ are sentences, then so are $\neg\sigma$ and $(\sigma \wedge \rho)$. If a sentence σ is true in a field K, we write

$$K \models \sigma;$$

and we define

$$\mathbf{Mod}(\sigma) = \{K \in \boldsymbol{C} : K \models \sigma\}.$$

Then

$$\begin{gathered}\mathbf{Mod}(\sigma_p) = \{K \in \boldsymbol{C} : \operatorname{char}(K) = p\},\\ \mathrm{Mod}(\neg\sigma) = \boldsymbol{C} \smallsetminus \mathbf{Mod}(\sigma),\\ \mathrm{Mod}(\sigma \wedge \rho) = \mathrm{Mod}(\sigma) \cap \mathrm{Mod}(\rho).\end{gathered}$$

If also Σ is a set of sentences, we define

$$\mathbf{Mod}(\Sigma) = \bigcup_{\sigma\in\Sigma} \mathbf{Mod}(\sigma);$$

this consists of the **models** of Σ. The classes $\mathbf{Mod}(\Sigma)$ are precisely the closed classes (we have to call them classes now) in a topology on $\boldsymbol{C}$. In this topology, if two fields have the same characteristic, then they belong to the same open classes. So we might as well consider the class of all fields having a given characteristic as being a single point. This is practically what we did in Example 14, in considering the set of *theories* of fields. The function

$$K \mapsto \mathrm{Th}(K)$$

from $\boldsymbol{C}$ to this set is continuous, and it induces a well-defined homeomorphism from $\boldsymbol{C}/{\sim}$ to the set, where

$$K \sim L \iff \mathrm{char}(K) = \mathrm{char}(L),$$

and $\boldsymbol{C}/{\sim}$ is given the *quotient topology,* defined generally as follows.

For an arbitrary topological space Ω, if $\sim$ is an equivalence relation on Ω, then the **quotient topology** on $\Omega/{\sim}$ is the finest in which the quotient map $x \mapsto [x]$ from Ω to $\Omega/{\sim}$ is continuous.

Example 22. The **torus** is $\mathbb{R}^2/{\sim}$, where

$$(a, b) \sim (x, y) \iff (a - x, b - y) \in \mathbb{Z}^2.$$

Suppose again we are given an indexed family $(\Omega_i : i \in I)$ of topological spaces, along with a set Ω. If for each i in I there is a function f_i from Ω to Ω_i, we define the **weak topology** on

$$\begin{tikzcd}
\Omega \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_{j \in I} \Omega_j \arrow[d, "\pi_i"] \\
& \Omega_i
\end{tikzcd}$$

Figure 2: An evaluation map

Ω to be the weakest in which the functions f_i, are continuous. In any case, given the f_i, we can define a function f from Ω to $\prod_{i \in I} \Omega_i$ by

$$f(x) = \big(f_i(x) \colon i \in I\big). \tag{31}$$

This function is the **evaluation map** for the f_i. It follows then that

$$f_i = \pi_i \circ f. \tag{32}$$

Conversely, if we are given f from Ω to $\prod_{j \in I} \Omega_j$, then, for each i in I, we can use (32) to define f_i from Ω to Ω_i; and in this case (31) holds. The functions f and f_i, related by (31) and (32), can be depicted as in Figure 2.

Theorem 13. *Let $(\Omega_i \colon i \in I)$ be an indexed family of topological spaces, let Ω be a topological space, and $f_i \colon \Omega \to \Omega_i$ for each i in I.*

1. *The evaluation map is continuous if and only if each f_i is continuous.*
2. *The evaluation map is injective if and only if the f_i **separate points** in the sense that, if $a \neq b$ in Ω, then, for some i in I, $f_i(a) \neq f_i(b)$.*

3. *The topology on Ω is at least as strong as the weak topology if and only if f is continuous.*
4. *The topology on Ω is the weak topology, and the f_i separate points, if and only if the evaluation map is an* ***embedding*** *in the sense of being a homeomorphism onto its image.*

Proof. Let the evaluation map be f. If this is continuous, then so are the f_i, since each π_i is continuous, and a composite of continuous functions is continuous. Suppose conversely each f_i is continuous. Then for each open subset U_i of Ω_i, the pre-image $f_i^{-1}[U_i]$ is open; but this is $f^{-1}\big[{\pi_i}^{-1}[U_i]\big]$, and sets ${\pi_i}^{-1}[U_i]$ compose a sub-basis for the topology on $\prod_{i\in I}\Omega_i$; so f must be continuous.

The next two claims are obviously true. Then in proving the last claim, we may assume f is injective, so that f^{-1} is well defined on $f[\Omega]$. We may assume also that f is continuous.

Suppose first f is an embedding. Let U be an open subset of Ω; we have to show that U is open in the weak topology. Since f^{-1} on $f[\Omega]$ is continuous by assumption, and

$$(f^{-1})^{-1}[U] = f[U],$$

this set is open in $f[\Omega]$. Then for some open subset V of $\prod_{i\in I}\Omega_i$,

$$f[U] = V \cap f[\Omega].$$

By this, and because f is injective, we have

$$U = f^{-1}\big[f[U]\big] = f^{-1}[V].$$

Thus U must already be open in the weak topology.

Now suppose conversely that Ω has the weak topology. Assuming U is an open subset of Ω, we have to show that $f[U]$

is open in $f[\Omega]$. But U is a union of finite intersections of sets ${f_i}^{-1}[V_i]$, where the V_i are open subsets of Ω_i. Also,

$$\begin{aligned} f\left[{f_i}^{-1}[V_i]\right] &= \{f(x)\colon f_i(x) \in V_i\} \\ &= \big\{(f_j(x)\colon j \in I)\colon f_i(x) \in V_i\big\} \\ &= \big\{(y_j\colon j \in I) \in f[\Omega]\colon y_i \in V_i\big\} \\ &= f[\Omega] \cap {\pi_i}^{-1}[V_i], \end{aligned}$$

and this is open in $f[\Omega]$. Since f is injective, so that the image under f of an intersection of sets is the intersection of the images of the sets, it follows that $f[U]$ itself is open in $f[\Omega]$. □

A quotient topology is an example of a *strong topology.* If Ω is a topological space, A is a set, and $f\colon \Omega \to A$, then the **strong topology** on A is the strongest in which f is continuous.

Theorem 14. *The strong topology on A as above consists of the subsets U of A such that $f^{-1}[U]$ is open in Ω.*

Proof. Let τ be the collection of such sets U. Then A itself is in τ. If τ contains U and V, then

$$f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V],$$

which is open, so $U \cap V \in \tau$. If $\{U_i\colon i \in I\} \subseteq \tau$, then

$$f^{-1}\left[\bigcup_{i \in I} U_i\right] = \bigcup_{i \in I} f^{-1}[U_i],$$

which is open, so $\bigcup_{i\in I} U_i \in \tau$. Thus τ is a topology on A. Immediately f is continuous with respect to τ, but not with respect to any stronger topology. □

$$\begin{tikzcd}
\Omega \arrow[r, "f"] \arrow[d, "\pi"'] & A \\
\Omega/{\sim} \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

Figure 3: A quotient

Given f from Ω to A, we can define on Ω the equivalence relation $\sim$ by

$$x \sim y \iff f(x) = f(y).$$

Then there is a well-defined function $\tilde{f}$ from $\Omega/{\sim}$ to A, and $\tilde{f} \circ \pi = f$ as in Figure 3, where π is $x \mapsto [x]$ from Ω to $\Omega/{\sim}$.

Theorem 15. *In the notation above, A having the strong topology, the strong topology on $\Omega/{\sim}$ with respect to π is the weak topology with respect to $\tilde{f}$.*

Proof. By Theorem 13, since $\tilde{f}$ is injective, it is an embedding with respect to the weak topology on $\Omega/{\sim}$. Thus the weakly open subsets of $\Omega/{\sim}$ are precisely the sets $\tilde{f}^{-1}[U]$, where U is an open subset of A. In this case $f^{-1}[U]$ is open by Theorem 14. But

$$f^{-1}[U] = \pi^{-1}\left[\tilde{f}^{-1}[U]\right],$$

so by Theorem 14 again, $\tilde{f}^{-1}[U]$ is strongly open.

Now let V be a strongly open subset of $\Omega/{\sim}$. Then $\pi^{-1}[V]$ is open. But

$$\pi^{-1}[V] \subseteq f^{-1}\Big[f\left[\pi^{-1}[V]\right]\Big]. \tag{33}$$

Conversely, if $a \in f^{-1}\Big[f\big[\pi^{-1}[V]\big]\Big]$, then $f(a) \in f\big[\pi^{-1}[V]\big]$, so for some b in $\pi^{-1}[V]$ we have $f(a) = f(b)$; but this just means $\pi(a) = \pi(b)$, so $a \in \pi^{-1}[V]$. Thus the inclusion (33) is an equation, so $f\big[\pi^{-1}[V]\big]$ is open. Moreover, since π is surjective and $\tilde{f}$ is injective, we can compute

$$\tilde{f}^{-1}\Big[f\big[\pi^{-1}[V]\big]\Big] = \tilde{f}^{-1}\Bigg[\tilde{f}\Big[\pi\big[\pi^{-1}[V]\big]\Big]\Bigg] = V.$$

Thus V is weakly open. □

9 Projective spaces

Example 23 (The projective plane). In Euclidean geometry, **Pappus's Theorem** is about a hexagon whose vertices lie alternately on two straight lines. There are three cases. Using Proposition I.27 of Euclid's *Elements,* that triangles on the same base and between the same parallels are equal, Pappus shows that, if each of two pairs of opposite sides of the hexagon are parallel, then the third pair are parallel. This means, in Figure 4, since ACE and BDF are straight, and $AB \parallel DE$, and $BC \parallel DE$, it follows that $CD \parallel AF$. With more work, involving **Thales's Theorem** (it is actually Proposition VI.2 of Euclid's *Elements*), Pappus shows that, if each pair of opposite sides of the parallelogram intersect, as in Figure 5, then the three intersection points are on a straight line. A third case is missing from Pappus's work: if one pair of opposite sides of the parallelogram are parallel, but another pair intersect, as in Figure 6, then the third pair intersect, and the straight line through the intersection points of the two intersecting pairs of opposite sides is parallel to the parallel pair. We can understand all three cases of the theorem as one, if we say that

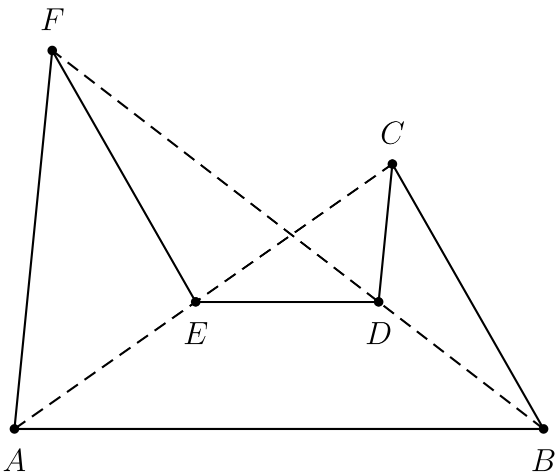

Figure 4: Pappus's Theorem with two pairs parallel

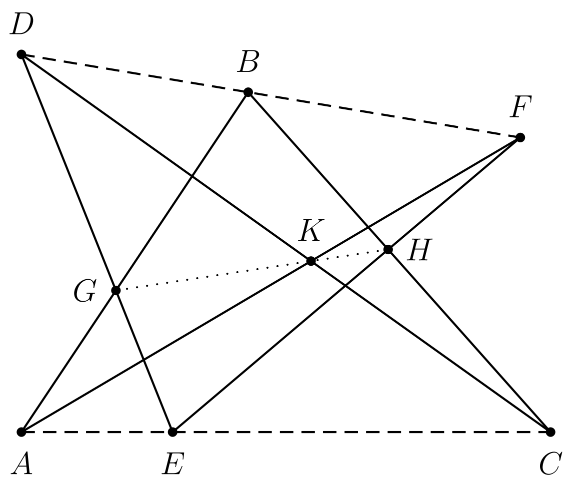

Figure 5: Pappus's Theorem with no pairs parallel

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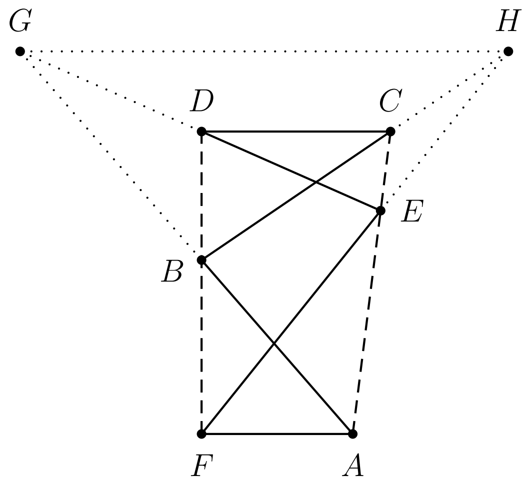

Figure 6: Pappus's Theorem with one pair not parallel

parallel straight lines meet at a point "at infinity." When to the Euclidean plane we introduce all of the needed points at infinity, we obtain the **projective plane.** We can do this as follows.

Parallelism in the plane is an equivalence relation. For each parallelism-class of straight lines in the plane, we need to introduce a new point. If we fix a point O in the plane, then each parallelism class has a unique member that passes through O. This is still true, even if we consider the plane as embedded in space, and O lies below the plane; and in this case, every straight line through O that is not parallel to the plane passes through the plane at a unique point. We now define the projective plane as the set of straight lines through a fixed point O in space. For each point A of space that is different from O, there is a unique straight line through A and O. This gives us a function onto the projective plane from the collection of points of space other than O. We can then give the projective plane the strong topology induced from the Euclidean topology of space.

Example 24 (Projective spaces)**.** There are likewise projective spaces of all finite dimensions. Given n in $\mathbb{N}$, we can define $\sim$ on $\mathbb{R}^{n+1} \smallsetminus \{\mathbf{0}\}$ so that $\boldsymbol{a} \sim \boldsymbol{b}$ if and only if, for some t in $\mathbb{R} \smallsetminus \{0\}$,

$$\boldsymbol{a} = t \cdot \boldsymbol{b}.$$

Then by definition

$$\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \smallsetminus \{\mathbf{0}\})/{\sim}.$$

This is **projective n-space.** The quotient map $\boldsymbol{x} \mapsto [\boldsymbol{x}]$ from $\mathbb{R}^{n+1} \smallsetminus \{\mathbf{0}\}$ onto $\mathbb{P}^n(\mathbb{R})$ is still surjective when restricted to the **n-sphere,** which defined by

$$\mathrm{S}^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \colon |\boldsymbol{x}| = 1\}.$$

Points $\boldsymbol{a}$ and $\boldsymbol{b}$ of the sphere are mapped to the same point of projective space if and only if $\boldsymbol{a} = \pm\boldsymbol{b}$. The function

$$\boldsymbol{x} \mapsto \frac{1}{|\boldsymbol{x}|} \cdot \boldsymbol{x}$$

from $\mathbb{R}^{n+1} \smallsetminus \{\mathbf{0}\}$ to itself is continuous, and for each i in $n+1$, it restricts to a homeomorphism from

$$\{\boldsymbol{x} \in \mathbb{R}^{n+1} \colon x_i = 1\},$$

which can be called a **hyperplane,** to

$$\{\boldsymbol{x} \in \mathrm{S}^n \colon x_i > 0\},$$

a hemisphere. The hyperplane here being homeomorphic with Euclidean n-space $\mathbb{R}^n$, we obtain an embedding of this space in projective n-space; and the points not in the image of the embedding make up a space homeomorphic with projective $(n-1)$-space (assuming $n > 0$).

10 Separation

If Ω is a topological space, and $(a_i : i \in I)$ is a sequence of points of Ω, then, by generalizing the usual definition from calculus, we say that the sequence **converges** to a, and a is a **limit** of the sequence, if for all neighborhoods N of a, for some m in ω, for all n in ω,

$$n \geqslant m \implies a_n \in N.$$

Theorem 16. *In a topological space, if any two distinct points of the space have disjoint neighborhoods, then limits of sequences are unique.*

A space satisfying the hypothesis of the theorem is called **Hausdorff** or T_2. Here the letter T stands for the German *Trennung* "separation." Every metric space is Hausdorff. In a Hausdorff space, the notation

$$\lim_{n\to\infty} a_n = a$$

is justified. But not every topological space is Hausdorff. A space is **Kolmogorov** or T_0 if, for any two distinct points, there is a neighborhood of one that does not contain the other. A space is T_1 if, for any two points, each of the points has a neighborhood that does not contain the other.

Example 25.
1. A trivial space (as in Example 8) with more than one element is not even Kolmogorov.
2. The Zariski topology in Example 13 (page 24) is Kolmogorov, but not T_1.
3. An infinite space with the cofinite topology (Example 9) is T_1, but not Hausdorff.

4. The Tarski topology in Example 14 is Hausdorff.

If a convergent sequence $(a_i : i \in \omega)$ is injective (that is, $a_i \neq a_j$ whenever $i \neq j$), or if it at least has infinitely many terms (that is, $\{a_i : i \in I\}$ is infinite), then every limit of the sequence is a limit point of the set of its terms.

Exercise 11. A subset A of a metric space is closed if, for every convergent sequence whose terms belong to a subset A, the limit belongs to A.

11 Countability

In an arbitrary metric space, a set may have a limit point that is not the limit of any sequence of points of the set. We shall establish this with Example 26 below. To do this, if A and B are two sets, we write

$$A \preccurlyeq B, \qquad A \approx B, \qquad A \prec B,$$

if there are, respectively, an injection from A to B, a bijection from A to B, and an injection, but no bijection. In the middle case, A and B are **equipollent.**

Theorem 17 (Cantor)**.** $A \prec \mathscr{P}(A)$.

Proof. There is an injection $x \mapsto \{x\}$ from A to $\mathscr{P}(A)$. Suppose f is a injection from A to $\mathscr{P}(A)$, and let $B = \{x \in A : x \notin f(x)\}$. For every c in A we have

$$c \in B \iff c \notin f(c);$$

this shows $B \neq f(c)$. Thus f cannot be a bijection. □

Theorem 18 (Schröder–Bernstein). *If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.*

Proof. Suppose f is an injection from A to B; and g, from B to A. We define recursively

$$A_0 = A \smallsetminus g[B], \qquad B_0 = B \smallsetminus f[A],$$
$$A_{n+1} = g[B_n], \qquad B_{n+1} = f[A_n].$$

Strictly, the definition requires us to observe at the same time by induction that

$$A_n \subseteq A, \qquad B_n \subseteq B.$$

We can conclude then

$$A_0 \cap A_{n+1} = \varnothing, \qquad B_0 \cap B_{n+1} = \varnothing.$$

By induction

$$A_k \cap A_{n+k+1} = \varnothing, \qquad B_k \cap B_{n+k+1} = \varnothing.$$

Also

$$A_k \approx B_{k+1}, \qquad A_{k+1} \approx B_k,$$

and so

$$A_k \cup A_{k+1} \approx B_k \cup B_{k+1}.$$

Therefore

$$\bigcup_{i \in \omega} A_i \approx \bigcup_{i \in \omega} B_i. \tag{34}$$

Finally, if we let

$$C_k = A \smallsetminus \bigcup_{i<k} A_i, \qquad D_k = B \smallsetminus \bigcup_{i<k} B_i,$$

then in each case

$$f[C_k] = D_{k+1},$$

so, since $D_0 \supseteq D_1$,

$$f\left[\bigcap_{i\in\omega} C_i\right] = \bigcap_{i\in\omega} D_i.$$

This with (34) gives $A \approx B$, since A is the disjoint union of $\bigcup_{i\in\omega} A_i$ and $\bigcap_{i\in\omega} C_i$, and similarly for B. □

Thus the relation $\prec$ determines a partial ordering of the classes of equipollent sets. By one formulation of the **Axiom of Choice** (see page 19), this ordering is a linear ordering. If $A \preccurlyeq \omega$, then A is called **countable.** The alternative is that A is **uncountable,** and this means $\omega \prec A$. By Cantor's Theorem, uncountable sets exist, $\mathscr{P}(\omega)$ being an example.

Theorem 19. $\mathbb{R} \approx \mathscr{P}(\omega)$.

Proof. We know from Example 7 (page 12) that $\mathscr{P}(\omega)$ is equipollent with the Cantor set, which is a subset of the interval $[0,1]$; so $\mathscr{P}(\omega) \preccurlyeq [0,1]$. We also have $[0,1] \preccurlyeq \mathscr{P}(\omega)$. Indeed, every element of the interval can be written uniquely as a sum

$$\sum_{i\in\omega} \frac{e_i}{2^{i+1}},$$

where $e_i \in 2$ and the sequence $(e_i \colon i \in \omega)$ is not eventually 0 unless every term is 0. Then the element of $[0,1]$ determines the subset $\{i \in \omega \colon e_i \neq 0\}$ of ω. By the Schröder–Bernstein Theorem, $\mathscr{P}(\omega) \approx [0,1]$; by Example 15 (page 30), $\mathbb{R} \approx (0,1)$, so we are done. □

Theorem 20. $\omega \times \omega \approx \omega$.

Proof. By the Schröder–Bernstein Theorem, it is enough to show $\omega \times \omega \preccurlyeq \omega$. This holds by means of the function

$$(k, m) \mapsto 2^k \cdot 3^m,$$

which is injective by the Fundamental Theorem of Arithmetic. $\square$

Corollary 1. *$\mathbb{Q}$ is countable.*

Corollary 2. *If $(A_i : i \in \omega)$ is a countable sequence of countable sets, then $\bigcup_{i\in\omega} A_i$ is countable.*

Proof. By the Axiom of Choice, for each i in ω, we can choose some injection f_i of A_i in ω. Then we can embed $\bigcup_{i\in\omega} A_i$ in $\omega \times \omega$ by sending an element a to $\bigl(k, f_k(a)\bigr)$, where $k = \min\{n \in \omega : a \in A_n\}$. $\square$

Every linearly ordered set Ω is a topological space, just as $\mathbb{R}$ is. There is a basis consisting of the open intervals, which are sets of one of the forms

$$\{x \in \Omega : a < x < b\}, \quad \{x \in \Omega : x < b\}, \quad \{x \in \Omega : a < x\};$$

we call these

$$(a, b), \qquad (-\infty, b), \qquad (a, \infty)$$

respectively. By yet another version of the **Axiom of Choice,** every set can be *well*-ordered.

Example 26. Let $\mathbb{R}$ be well-ordered by a relation $\vartriangleleft$, and let Ω consist of those a in $\mathbb{R}$ such that $\{x \in \mathbb{R} : x \vartriangleleft a\}$ is countable. If A is a subset of Ω, let

$$A^* = \bigcup_{x\in A} \{y \in \mathbb{R} : y \vartriangleleft x\}.$$

If A is countable, then so is A^*, by the second corollary of Theorem 20. In this case, $\mathbb{R} \smallsetminus A^* \neq \varnothing$, so we can let b be the least real number with respect to $\lhd$ that is *not* in A^*. Then

$$A^* = \{x \in \mathbb{R} \colon x \lhd b\},$$

so $b \in \Omega$. This shows that every countable subset of Ω has a strict upper bound in Ω. In particular, Ω has no greatest element and is itself uncountable. Now define ∞ to be greater than all elements of Ω. Then ∞ cannot be a limit point of any countable subset of Ω. In particular, no countable sequence in Ω converges to ∞. But ∞ is a limit point of Ω. If $a \in \Omega$, we understand $(-\infty, a)$ as $\{x \in \Omega \colon x \lhd a\}$. Since Ω has no greatest element, we have

$$\bigcup_{x \in B} (-\infty, a) = \Omega$$

when B is all of Ω, but never when B is a countable subset.

In the example, as a ranges over Ω, the intervals $(-\infty, a)$ constitute an **open covering** of Ω; however, there is no countable sub-covering.

Theorem 21 (Lindelöf Covering Theorem)**.** *For every open covering of a subset of* $\mathbb{R}$*, there is a countable sub-covering.*

Proof. The topology of $\mathbb{R}$ has basis consisting of the open intervals (a, b), where a and b are in $\mathbb{Q}$. In particular, by the first corollary of Theorem 20, $\mathbb{R}$ has a countable base $\mathscr{B}$. Suppose $\mathscr{O}$ is an open covering of a subset A of $\mathbb{R}$. For each a in A, there are U_a in $\mathscr{B}$ and O_a in $\mathscr{O}$ such that

$$a \in U_a, \qquad\qquad U_a \subseteq O_a.$$

Let $\mathscr{C}$ be the subcollection $\{U_x : x \in \mathbb{R}\}$ of $\mathscr{B}$. Then $\mathscr{C}$ is countable, and $\bigcup \mathscr{C} \supseteq A$. Moreover, for each U in $\mathscr{C}$, by the Axiom of Choice, there is U^* in $\mathscr{O}$ such that $U \subseteq U^*$. Then $\{X^* : X \in \mathscr{C}\}$ is a countable subcollection of $\mathscr{O}$ whose union includes A. □

We consider some possible properties of spaces that involve countability. A **neighborhood base** of a point in a topological space is a collection of neighborhoods of the point in which can be found a subset of every neighborhood of the point. A subset of a space is **dense** if its closure is the whole space. We now say that a space is

1) **first-countable,** if its every point has a countable neighborhood base;
2) **second-countable,** if the space itself has a countable base;
3) **separable,** if it has a countable dense subset;
4) **Lindelöf,** if for every open covering of the space, there is a countable sub-covering.

Example 27. $\mathbb{R}$ has all four of the properties just named. In particular, $\mathbb{Q}$ is dense in $\mathbb{R}$. An uncountable discrete space has only the first property; the same is true for Ω in Example 26.

Exercise 12. Every second-countable space is first-countable, separable, and Lindelöf.

Example 28. The subsets $[a, b)$ of $\mathbb{R}$, where $a < b$, compose a basis for a topology, since, if $c < b$, then

$$[a, b) \cap [c, d) = \bigl[\max\{a, c\}, \min\{b, d\}\bigr),$$

and every point of $\mathbb{R}$ belongs to some interval $[a, b)$. We shall denote the resulting space by $\mathbb{E}$: it is the **Sorgenfrey line.**

We can show that it is Lindelöf as follows. Suppose a collection $\{[a_i, b_i)\colon i \in A\}$ of intervals covers $\mathbb{R}$. Let

$$U = \bigcup\{(a_i, b_i)\colon i \in A\}.$$

By the Lindelöf Covering Theorem, A has a countable subset B_0 such that

$$U = \bigcup\{(a_i, b_i)\colon i \in B_0\}.$$

If $U = \mathbb{R}$, we are done; but possibly $U \subset \mathbb{R}$. For every point c of $\{-\infty\} \cup (\mathbb{R} \smallsetminus U)$, we can define

$$f(c) = \inf\bigl((c, \infty) \smallsetminus U\bigr),$$

where $\inf(\varnothing) = \infty$. For some i in B_0, we have

$$c \in [a_i, b_i) \smallsetminus (a_i, b_i).$$

Then $c = a_i < b_i \leqslant f(c)$. Thus $\bigl(c, f(c)\bigr)$ is an interval and is included in U. For every d in U, since U is open in the Euclidean topology, the point $\sup\bigl((-\infty, d) \smallsetminus U\bigr)$, where $\sup(\varnothing) = -\infty$, must actually be an element c of $\{-\infty\} \cup (\mathbb{R} \smallsetminus U)$, and then $d \in (c, f(c))$. Thus

$$U = \bigcup_{x \in \{-\infty\} \cup \mathbb{R} \smallsetminus U} \bigl(x, f(x)\bigr).$$

Since these intervals are disjoint, we can conclude, again by the Lindelöf Covering Theorem, that $\mathbb{R} \smallsetminus U$ is countable. Thus for some countable subset B_1 of A,

$$\mathbb{R} \smallsetminus U \subseteq \bigcup\bigl\{[a_i, b_i)\colon i \in B_1\bigr\},$$
$$\mathbb{R} = \bigcup\bigl\{[a_i, b_i)\colon i \in B_0 \cup B_1\bigr\}.$$

Since $B_0 \cup B_1$ is countable, $\mathbb{E}$ must be Lindelöf.

Exercise 13. Show that $\mathbb{E}$ is (a) first-countable, and (b) separable, but (c) not second-countable.

Exercise 14. Let Ω be an uncountable set with a particular element a. Show that

(a) Ω has a topology in which the open sets are precisely the subsets of $\Omega \smallsetminus \{a\}$ and the complements of finite subsets of this; also, this topology is

(b) Lindelöf, but

(c) not first-countable and

(d) not separable.

12 Compactness

Theorem 22. *For every open covering of a closed bounded subset of $\mathbb{R}$ there is a finite sub-covering.*

Proof. Let F be a closed bounded subset of $\mathbb{R}$. By the Lindelöf Covering Theorem, any open covering of F has a sub-collection $\{O_i : i \in \omega\}$ whose union also includes F. Then we have a decreasing sequence

$$F \supseteq F \smallsetminus O_0 \supseteq F \smallsetminus (O_0 \cup O_1) \supseteq F \smallsetminus (O_0 \cup O_1 \cup O_2) \supseteq \cdots$$

of closed bounded subsets of $\mathbb{R}$ whose intersection is empty. By the contrapositive form of the Cantor Intersection Theorem (Theorem 12), the sequence has an empty term. This means that, for some n in ω,

$$F \subseteq \bigcup_{i<n} O_i. \qquad \square$$

In a word, every closed bounded subset of $\mathbb{R}$ is *compact.* In general, a topological space is **compact** if for every open covering of the space, there is a finite sub-covering. When we speak of a subset of a space as being compact, we mean that it is compact in the subspace topology.

Example 29. A closed subset F of a compact space is compact, since if $\mathscr{O}$ is an open covering of F, then $\mathscr{O} \cup \{F^{\mathrm{c}}\}$ is an open covering of the whole space, and so there is a finite subcovering $\mathscr{N}$, and then $\mathscr{N} \smallsetminus \{F^{\mathrm{c}}\}$ covers F and is a finite sub-collection of $\mathscr{O}$.

Exercise 15. The image of a compact space under a continuous function is compact.

Theorem 23. *The product of two compact spaces is compact.*

We now have the following corollary of the last two theorems.

Theorem 24 (Heine–Borel)**.** *For every n in $\mathbb{N}$, every closed and bounded subset of $\mathbb{R}^n$ is compact.*

A collection of sets has the **finite-intersection property** if every finite sub-collection has nonempty intersection.

Theorem 25. *A space is compact if and only if every collection of closed subsets with the finite-intersection property has nonempty intersection.*

Proof. Let Ω be a topological space. If $\mathscr{F}$ is a collection of closed subsets of Ω with the finite intersection property, then $\{X^{\mathrm{c}} \colon X \in \mathscr{F}\}$ is a collection $\mathscr{O}$ of open subsets of Ω, and no finite subset of $\mathscr{O}$ covers Ω. If Ω is compact, then $\mathscr{O}$

itself must not cover Ω, and this means $\bigcap \mathscr{F} \neq \varnothing$. Conversely, if $\mathscr{O}$ is an open covering of Ω, let $\mathscr{F}$ be the collection $\{X^{\mathrm{c}} \colon X \in \mathscr{O}\}$ of closed subsets of Ω. Then $\bigcap \mathscr{F} = \varnothing$. If every collection of closed subsets with the finite intersection property has nonempty intersection, then $\mathscr{F}$ must not have the finite-intersection property. In this case, some finite subset of $\mathscr{O}$ covers Ω. □

Exercise 16. Using the idea of Example 18, we have the converse of the Heine–Borel Theorem.

Theorem 26 (Bolzano–Weierstrass)**.** *For every n in $\mathbb{N}$, every bounded infinite subset of $\mathbb{R}^n$ has a limit point.*

Proof. Let A be a bounded subset of $\mathbb{R}^n$. If A is not closed, then it must have a limit point. Suppose A is closed, but has no limit points. Then every point of A has an open neighborhood that contains no other point of A. These open neighborhoods constitute an open covering of A for which there is no subcovering. Since A is compact by the Heine–Borel Theorem, A must be finite. □

Example 30 (Divide and Conquer)**.** One can also prove the Bolzano–Weierstrass Theorem independently. We do this in $\mathbb{R}$. Suppose A is an infinite subset of a closed bounded interval $[a, b]$, and let $c = (a+b)/2$. Then A has infinitely many points in at least one of the intervals $[a, c]$ and $[c, b]$. Let $[a^*, b^*]$ be one of these intervals so that $A \cap [a^*, b^*]$ is infinite. We now define recursively

$$[a_0, b_0] = [a, b], \qquad\qquad [a_{n+1}, b_{n+1}] = [{a_n}^*, {b_n}^*].$$

By the Cantor Intersection Theorem, the intervals $[a_n, b_n]$ have a common point d. For every positive r in $\mathbb{R}$, there is n in ω

such that $b_n - a_n < r$. In this case, $[a_n, b_n] \subseteq \mathrm{B}(d; r)$, so the ball contains infinitely many elements of A. Thus d is a limit point of A.

Example 31 (Logical compactness). We can prove the compactness of the Tychonoff topology on $\mathscr{P}(\omega)$ by another "Divide and Conquer" method. We use the setting of Example 20 (page 35), where the closed subsets of $\mathscr{P}(\omega)$ are just the sets of models of formulas and collections of formulas. Given a collection $\mathscr{F}$ of formulas whose every finite subset has a model, we show that $\mathscr{F}$ has a model. The assumption is that $\{\mathrm{Mod}(F)\colon F \in \mathscr{F}\}$ has the finite-intersection property; we aim to prove that the whole collection has nonempty intersection. We first observe that, for all formulas H, one of the collections

$$\{\mathrm{Mod}(F)\colon F \in \mathscr{F}\} \cup \{\mathrm{Mod}(H)\},$$
$$\{\mathrm{Mod}(F)\colon F \in \mathscr{F}\} \cup \{\mathrm{Mod}(\neg H)\}$$

has the finite-intersection property. For suppose the former does not. Then $\mathscr{F}$ has a finite subset $\{F_i\colon i < m\}$, in every model of which, $\neg H$ is true. Then for every finite subset $\{G_j\colon j < n\}$ of $\mathscr{F}$, the set $\{G_j\colon j < n\} \cup \{\neg H\}$ has models, namely the models of $\{G_j\colon j < n\} \cup \{F_i\colon i < m\}$, which exist by assumption. Thus $\{\mathrm{Mod}(F)\colon F \in \mathscr{F}\} \cup \{\mathrm{Mod}(\neg H)\}$ has the finite intersection property.

There is now a sequence $(H_i\colon i \in \omega)$ where each H_i is either P_i or $\neg P_i$, and for each n in ω, the collection

$$\{\mathrm{Mod}(F)\colon F \in \mathscr{F}\} \cup \{\mathrm{Mod}(H_i)\colon i < n\}$$

has the finite intersection property. Now let A be the subset of ω consisting of those n such that H_n is P_n. This means A is the unique model of $\{H_n : n \in \omega\}$. Let $F \in \mathscr{F}$. For some n in ω, for every propositional variable P_k appearing in F, $k < n$. Then for all B and C in $\mathscr{P}(\omega)$,

$$B \cap n = C \cap n \implies F^B = F^C.$$

We know that $\mathrm{Mod}(\{F\} \cup \{H_i : i < n\})$ is nonempty. If it contains B, then $B \cap n = A \cap n$, so $A \in \mathrm{Mod}(F)$. Thus A is a model of $\mathscr{F}$.