MSGSÜ, MAT 221

Final Sınavı çözümleri

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Notlandıran için çözümlerinizin nasıl okunacağı açık olmalı.
İngilizceyi veya Türkçeyi kullanabilirsiniz.

Problem 1. Let p be prime and $0 < k < p$. We know $k! \cdot (p-k)! \mid p!$. Show that

$$p \left| \frac{p!}{k! \cdot (p-k)!} \right. .$$

Solution. We are given that $p! = a \cdot k! \cdot (p-k)!$ for some a. We want to show $p \mid a$. We know $p \mid p!$, that is,

$$p \mid a \cdot k! \cdot (p-k)!.$$

But $p \nmid k!$ and $p \nmid (p-k)!$. Therefore, by Euclid's Lemma, $p \mid a$.

Problem 2. Assume a is an odd integer. For all natural numbers k, show

$$a^{2^k} \equiv 1 \pmod{2^{k+2}}.$$

Solution. We use induction.

1. When $k = 1$, the claim is $a^2 \equiv 1 \pmod 8$ for all odd a. This claim is true, since $(\pm 1)^2 \equiv 1$ and $(\pm 3) \equiv 1 \pmod 8$.
2. Suppose the claim is true when $k = \ell$. We want to show $2^{\ell+3} \mid a^{2^{\ell+1}} - 1$. We have

$$a^{2^{\ell+1}} - 1 = \left(a^{2^\ell}\right)^2 - 1 = (a^{2^\ell} + 1)(a^{2^\ell} - 1).$$

By inductive hypothesis, $2^{\ell+2} \mid a^{2^\ell} - 1$. Since a is odd, $2 \mid a^{2^\ell} + 1$. Therefore $2 \cdot 2^{\ell+2} \mid a^{2^{\ell+1}} - 1$.

Problem 3. Find the least positive integer x such that

$$x \equiv 10 \pmod{20} \qquad \& \qquad x \equiv 3 \pmod{23}.$$

Solution. First apply the Euclidean algorithm to 20 and 23:

$$\begin{aligned} 23 &= 20 + 3, & 1 &= 3 - 2 \\ 20 &= 3 \cdot 6 + 2, & &= 3 - (20 - 3 \cdot 6) = 3 \cdot 7 - 20 \\ 3 &= 2 \cdot 1 + 1, & &= (23 - 20) \cdot 7 - 20 = 23 \cdot 7 - 20 \cdot 8. \end{aligned}$$

Then

$$\begin{aligned} x &\equiv 10 \cdot 23 \cdot 7 - 3 \cdot 20 \cdot 8 = 1610 - 480 \\ &= 1130 = 460 \cdot 2 + 210 \equiv 210 \pmod{460}. \end{aligned}$$

Thus $x = 210$.

Problem 4. Find the least positive integer x such that

$$7^{10\,000\,002} \equiv x \pmod{1375}.$$

Solution. $1375 = 5 \cdot 275 = 5^2 \cdot 55 = 5^3 \cdot 11$, so

$$\phi(1375) = 5^3 \cdot 11 \cdot \frac{4}{5} \cdot \frac{10}{11} = 5^3 \cdot 2^3 = 1000.$$

By Euler's Theorem, $7^{1000} \equiv 1 \pmod{1375}$. Since also

$$10\,000\,002 \equiv 2 \pmod{1000},$$

we have $7^{10\,000\,002} \equiv 7^2 \pmod{1375}$, and thus $x = 49$.