MSGSÜ, MAT 221

Bütünleme Sınavı

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19 Ocak 2015, Saat 13:30

Notlandıran için çözümlerinizin nasıl okunacağı açık olmalı.
İngilizceyi veya Türkçeyi kullanabilirsiniz.

Problem 1. Find the least positive integer x such that

$$3^{99995} \cdot x \equiv 1 \pmod{27500}.$$

Solution. We have $27500 = 2^2 \cdot 5^2 \cdot 275 = 2^2 \cdot 5^3 \cdot 55 = 2^2 \cdot 5^4 \cdot 11$, so

$$\phi(27500) = 2^2 \cdot 5^4 \cdot 11 \cdot \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{10}{11} = 2^4 \cdot 5^4 = 10000.$$

Since $\gcd(3, 27500) = 1$, we have $3^{10000} \equiv 1 \pmod{27500}$, by Euler's Theorem; also, *modulo* 27500,

$$\begin{aligned} & 3^{99995} \cdot x \equiv 1 \\ \iff & 3^{100000} \cdot x \equiv 3^5 \\ \iff & (3^{10000})^{10} \cdot x \equiv 3^5 \\ \iff & x \equiv 3^5. \end{aligned}$$

Thus $x = 3^5 = 243$ (this is the least positive integer that is congruent to 3^5 *modulo* 27500).

Problem 2. Show that, for all primes p and all natural numbers k and ℓ,

$$\sigma(p^{k+\ell}) \leqslant \sigma(p^k) \cdot \sigma(p^\ell).$$

(Since σ is multiplicative, this shows that, for arbitrary natural numbers a and b, $\sigma(ab) \leqslant \sigma(a) \cdot \sigma(b)$.)

Solution. Since $\sigma(p^k) = (p^{k+1} - 1)/(p - 1)$, it is enough to show

$$\frac{p^{k+\ell+1} - 1}{p - 1} \leqslant \frac{p^{k+1} - 1}{p - 1} \cdot \frac{p^{\ell+1} - 1}{p - 1},$$

or equivalently

$$\begin{gathered}
(p^{k+\ell+1} - 1)(p - 1) \leqslant (p^{k+1} - 1)(p^{\ell+1} - 1), \\
p^{k+\ell+2} - p^{k+\ell+1} - p + 1 \leqslant p^{k+\ell+2} - p^{\ell+1} - p^{k+1} + 1, \\
p^{\ell+1} - p \leqslant p^{k+\ell+1} - p^{k+1}, \\
1 \leqslant p^k.
\end{gathered}$$

Since indeed $1 \leqslant p^k$, we must have $\sigma(p^{k+\ell}) \leqslant \sigma(p^k) \cdot \sigma(p^\ell)$..

Problem 3. Suppose p is prime, and r is such that, for some m such that $p \nmid m$,

$$r^{p-1} = 1 + m \cdot p.$$

Show that, for every natural number k, for some m such that $p \nmid m$,

$$r^{p^{k-1} \cdot (p-1)} = 1 + m \cdot p^k.$$

You may use that $p \mid \binom{p}{j}$ whenever $0 < j < p$.

Solution. We use induction. We are given that the claim is true when $k = 1$. Suppose it is true when $k = \ell$, so that, for some n such that $p \nmid n$,

$$r^{p^{\ell-1} \cdot (p-1)} = 1 + n \cdot p^\ell.$$

Then

$$\begin{aligned}
r^{p^\ell\cdot(p-1)} &= (1+n\cdot p^\ell)^p \\
&= 1 + n\cdot p\cdot p^\ell + n^2\cdot\binom{p}{2}\cdot p^{2\ell} + \cdots + n^p\cdot p^{p\ell} \\
&= 1 + \left(n + \underbrace{n^2\cdot\binom{p}{2}\cdot p^{\ell-1} + \cdots + p^{(p-1)\cdot\ell-1}}_{s}\right)\cdot p^{\ell+1} \\
&= 1 + (n+s)\cdot p^{\ell+1},
\end{aligned}$$

where $p \mid s$, so $p \nmid n+s$. Thus the claim is true when $k = \ell+1$, assuming it is true when $k = \ell$. By induction, the claim is true for all k.

Problem 4. Find the least positive integer x such that

$$x \equiv 2 \pmod{34} \qquad \& \qquad x \equiv 1 \pmod{21}.$$

Solution. First apply the Euclidean algorithm to 34 and 21:

$$\begin{aligned}
34 &= 21 + 13, & 1 &= 3 - 2 \\
21 &= 13 + 8, & &= 3 - (5-3) = 3\cdot 2 - 5 \\
13 &= 8 + 5, & &= (8-5)\cdot 2 - 5 = 8\cdot 2 - 5\cdot 3 \\
8 &= 5 + 3, & &= 8\cdot 2 - (13-8)\cdot 3 = 8\cdot 5 - 13\cdot 3 \\
5 &= 3 + 2, & &= (21-13)\cdot 5 - 13\cdot 3 = 21\cdot 5 - 13\cdot 8 \\
3 &= 2 + 1, & &= 21\cdot 5 - (34-21)\cdot 8 = 21\cdot 13 - 34\cdot 8.
\end{aligned}$$

Then

$$\begin{aligned}
x &\equiv 2\cdot 21\cdot 13 - 34\cdot 8 \\
&\equiv 21\cdot 13 + 21\cdot 13 - 34\cdot 8 \\
&\equiv 21\cdot 13 + 1 \pmod{34\cdot 21},
\end{aligned}$$

and therefore $x = 21\cdot 13 + 1 = 274$. This is easily checked: immediately $21\cdot 13 + 1 \equiv 1 \pmod{21}$; also $34\cdot 8 = 272$, so $274 \equiv 2 \pmod{34}$.