# Exam solutions

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Number Theory in English (MAT 221)
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**Problem 1.** *What is wrong with the following argument?*

1. *All positive integers are odd.*
2. *We shall prove this by mathematical induction.*
3. *Obviously* $1$ *is odd.*
4. *As an inductive hypothesis, we suppose that* $1$, $2$, …, $n-1$, $n$ *are all odd.*
5. *If* $n+1$ *is odd, we are done.*
6. *Suppose* $n+1$ *is even.*
7. *We have* $n+1 \equiv n-1 \pmod{2}$.
8. *By inductive hypothesis,* $n-1$ *is odd.*
9. *Therefore* $n+1$ *is odd.*
10. *By induction, all positive integers are odd.*

**Solution.** Step 8 is wrong, since $n$ could be 1, and in this case $n-1$ is not covered by the inductive hypothesis.

*Remark.* This is the only correct answer. Step 8 is wrong, and Step 8 is wrong *because* the set $\{1, 2, \dots, n-1, n\}$ does not actually contain $n-1$

if $n = 1$. (The reason why Step 8 is wrong is not that $n - 1$ is not always odd.) Step 1 contains an incorrect statement, but it is not an error in the argument as a whole. (If the argument in Steps 2–10 were correct, then Step 1 would be correct.) Step 4 makes the inductive hypothesis that all elements of the set $\{x \in \mathbb{Z}\colon 1 \leqslant x \leqslant n\}$ are odd. Steps 5 and 6 are not actually needed for the argument, but they are not incorrect. Step 7 contains an obviously correct statement. If Step 8 were correct, then Step 9 would be correct.

**Problem 2.** *Recall that the triangular numbers are defined recursively by*

$$t_1 = 1, \qquad\qquad t_{n+1} = t_n + n + 1.$$

*Prove by induction that, for all positive integers $n$,*

$$t_n + t_{n+1} = (n+1)^2.$$

**Solution.** We have

$$t_1 + t_2 = 2t_2 + 2 = 4 = 2^2.$$

Thus the claim holds when $n = 1$. Suppose the claim holds when $n = k$, that is,

$$t_k + t_{k+1} = (k+1)^2.$$

Then

$$\begin{aligned} t_{k+1} + t_{k+2} &= t_k + k + 1 + t_{k+1} + k + 2 \\ &= t_k + t_{k+1} + 2k + 3 \\ &= (k+1)^2 + 2k + 3 \\ &= k^2 + 2k + 1 + 2k + 3 \\ &= k^2 + 4k + 4 \\ &= (k+2)^2. \end{aligned}$$

Thus the claim holds when $n = k + 1$. Therefore, by induction, the claim holds for all positive integers $n$.

*Remark.* The claim can be proved *directly* (without induction) if one knows $t_n = (t+1)t/2$. However, the problem does not provide this information. When one uses induction, one should *not* present the argument as follows:

$$t_{k+1} + t_{k+2} \stackrel{?}{=} (k+2)^2$$
$$t_k + k + 1 + t_{k+1} + k + 2 \stackrel{?}{=} k^2 + 4k + 4$$
$$t_k + t_{k+1} \stackrel{?}{=} k^2 + 2k + 1$$
$$t_k + t_{k+1} = (k+1)^2$$

*Do not write this way.* Why not? Because nothing here is known to be correct, except the last line. The logical connection between the lines is not clear. One should rearrange the lines and write

$$t_k + t_{k+1} = (k+1)^2 = k^2 + 2k + 1,$$

therefore

$$t_k + k + 1 + t_{k+1} + k + 2 = k^2 + 4k + 4,$$

that is,

$$t_{k+1} + t_{k+2} = (k+2)^2.$$

**Problem 3.** *The following is a variant of a famous problem (discussed in class) from the ancient Chinese work called* Mathematical Classic of Master Sun.

> *Now there are an unknown number of things. If we count by threes, there is a remainder* 1*; if we count by fives, there is a remainder* 2*; if we count by sevens, there is a remainder* 3*. Find the number of things.*

**Solution.** We have to solve

$$x \equiv 1 \pmod 3, \qquad x \equiv 2 \pmod 5, \qquad x \equiv 3 \pmod 7.$$

The moduli 3, 5, and 7 are pairwise coprime. (In fact they are all distinct primes.) We compute some inverses:

$$5 \cdot 7 = 35 \equiv 2 \pmod 3,$$
$$2 \cdot 2 \equiv 1 \pmod 3,$$
$$3 \cdot 7 = 21 \equiv 1 \pmod 5,$$
$$3 \cdot 5 = 15 \equiv 1 \pmod 7.$$

Also $3 \cdot 5 \cdot 7 = 21 \cdot 5 = 105$. Therefore

$$x \equiv 35 \cdot 2 + 2 \cdot 21 + 3 \cdot 15 = 70 + 42 + 45 = 157 \equiv 52 \pmod{105}.$$

The number of things is 52, or $52 + 105n$ for some positive integer $n$.

**Problem 4.** *Compute* $9^{1815}$ modulo 19. *That is, find all integers $x$ such that*

$$0 \leqslant x < 19 \qquad \textit{and} \qquad 19 \mid 9^{1815} - x.$$

**Solution.** Since 19 is prime, by Fermat's Theorem we have, *modulo* 19,

$$\begin{aligned} 9^{1815} &\equiv (9^{18})^{100} \cdot 9^{15} \equiv 9^{15} = 9^{8+4+2+1} = 9^8 \cdot 9^4 \cdot 9^2 \cdot 9, \\ 9^2 &= 81 \equiv 5, \\ 9^4 &\equiv 5^2 = 25 \equiv 6, \\ 9^8 &\equiv 6^2 = 36 \equiv -2, \\ 9^{1815} &\equiv -2 \cdot 6 \cdot 5 \cdot 9 = -18 \cdot 30 \equiv 30 \equiv 11. \end{aligned}$$

*Remark.* I believe the foregoing solution is the most efficient. It is less efficient to compute, for example,

$$9^{1815} \equiv (9^{19})^{95} \cdot 9^{10} \equiv 9^{95} \cdot 9^{10} \equiv (9^{19})^5 \cdot 9^{10} \equiv 9^{15}.$$

In any case, one must not remember Fermat's Theorem incorrectly.

**Problem 5.** *Solve the congruence*

$$12x \equiv 6 \pmod{18},$$

*that is, find all integers $k$ such that*

$$12k \equiv 6 \pmod{15} \qquad \textit{and} \qquad 0 \leqslant k \leqslant 15.$$

**Solution.** Because $\gcd(12, 6) = 6$ and $\gcd(6, 15) = 3$, we have

$$\begin{aligned} 12x \equiv 6 \pmod{15} &\iff 2x \equiv 1 \pmod 5 \\ &\iff 6x \equiv 3 \pmod 5 \\ &\iff x \equiv 3 \pmod 5 \\ &\iff x \equiv 3, 8, 13 \pmod{15}. \end{aligned}$$

**Bonus.** *List the odd primes in increasing order:*

$$3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ \ldots$$

*Prove that the sum of any two consecutive numbers on this infinite list is the product of three integers that are greater than* 1*. For example,*

$$3+5 = 2\cdot 2\cdot 2, \qquad 17+19 = 3\cdot 3\cdot 4, \qquad 19+23 = 2\cdot 3\cdot 7.$$

**Solution.** Suppose $p$ and $q$ are consecutive odd primes. Then $p+q$ is even, and

$$p < \frac{p+q}{2} < q.$$

Therefore $(p+q)/2$ is composite: say $(p+q)/2 = ab$, where $a$ and $b$ are both greater than 1. Then

$$p+q = 2ab.$$