Ordinal Analysis

David Pierce

June 6, 2017
Revised May 29, 2018

Matematics Department
Mimar Sinan Fine Arts University
Istanbul
http://mat.msgsu.edu.tr/~dpierce/

Preface

This essay began as the preface of a text for the course called *Aksiyomatik Kümeler Kuramı* (Axiomatic Set Theory) in my department in Istanbul. Written in Turkish, the text is based on my lectures in the spring semester of 2015–6.

Contents

1 Introduction

In an undergraduate course, I propose to develop set theory on the model of calculus, by likening the ordered class of ordinals to the ordered set of real numbers, which should already be familiar to the students.

One may read the present essay from back to front.

According to **§7,** students of set theory should learn to (i) distinguish classes from sets, (ii) add and multiply ordinals using Cantor normal forms, (iii) recognize and prove ordinal identities, (iv) supply counterexamples to false identities and find errors in false proofs, (v) perform cardinal computations using Alephs and Beths.

I review my own textbook of these things in **§6,** chapter by chapter.

In **§5** I describe my habit of writing texts for courses. In the past I tried to write down everything the *teacher* (myself) should know or might want to know. The student had to understand from class what he or she should know. To prevent any confusion, I have now tried to strip my set-theory text down to the essentials.

Mathematics has a notion of *correctness.* We must conform to standards. What students of mathematics must learn, and all they need learn, is that the standards of mathematics are within themselves, although they are not simply personal. I review these ideas in **§4,** looking at Henry David Thoreau as somebody who tried to play life by his own rules. I distinguish the universality of logic from the variability of definitions and conventions; here I consider the example of the intersection of the empty set, which is sometimes left undefined, or defined as the empty set, but for me is the universal class.

For a first course that can really teach how mathematics is *hypothetical,* rather than simply empirical, I suggest in **§2** that set theory is the best bet. The propositions of Euclidean geometry may be too obviously true to be understood as hypothetical, while non-Euclidean geometry can be too much of a challenge to the intuition. The hypotheses of set theory can *build* on intuitions already developed in calculus.

The notion of an hypothetical science did not issue fully formed from the tongue of Thales or the pen of Euclid. Thales may have discovered the possibility of mathematical proof. Plato recognized that mathematics used hypotheses, but these were "the odd and the even and the various figures and three kinds of angles and other things akin to these": apparently not such grand assertions as Euclid's Parallel Postulate, or Archimedes's postulate on convex curves. I review these things in **§3.**

For Plato, hypotheses are to be "done away with" through dialectic. Under a positivistic interpretation, this means establishing them empirically. R. G. Collingwood is closer to the mark: the aim is to *stop* making certain hypotheses, if dialectic has exposed them as untenable. The importance of this is clear when we consider the hypotheses that humans are classified naturally by race and sex, and there is no notion of "gender" distinct from sex.

In mathematics, we test our hypotheses by seeing what we can prove from them, *and* by considering whether our hypotheses might not be theorems derived from simpler hypotheses. Set theory lets us do this for analysis. Abraham Robinson's so-called "non-standard" analysis gives us *back* the hypothesis of infinitesimals that was once removed as non-rigorous. An undergraduate course of set theory may not go so far, but can at least establish the mathematical reality of the infinite and the uncountable.

2 Mathematics as such

I propose to distinguish between two notions of mathematics: between (1) mathematics as a natural science and (2) mathematics as a logical system. A course of set theory may help to make the distinction and to carry the student from the one to the other.

Arnol′d [2] comments on mathematics in the first sense:

> Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.

As a natural science, mathematics should agree with the world and thus with other mathematics courses; in particular, whatever students have learned in high school, they should be able to rely on and use at university. In actual fact, students are sometimes misguided by their earlier learning. If in high school they are taught the Sarrus Rule for computing determinants of 3×3 matrices, at university they may apply a similar rule to a 4×4 matrix. In attempting this generalization, students might then be described as using the inductive logic of a natural science; but they get wrong answers.

In §4 I shall consider another example: having seen $\bigcap \varnothing$ defined as $\varnothing$ in one source, one student failed to see that, from the general definition of the intersection of a class, $\bigcap \varnothing$ should be the universal class.

Treated logically, mathematics is *hypothetical*: it derives from so-called axioms, which may be plausible, but will in any case be accepted within a particular course, though perhaps not in others. The question then is not what is true, but of what follows truly from the axioms. There may be the further question of which rules of inference can be applied to the axioms; in practice, this question does not often arise, unless one wants to avoid proofs by contradiction.

Speaking to us from 2300 years ago, Euclid recognizes the gap between the two conceptions of mathematics. He has given us in the *Elements* [13] the earliest extant sustained attempt to bridge the gap. This is not to say that the gap itself may not have been recognized earlier, in Egypt, Babylonia, India, or China.

In *The Critique of Pure Reason* [18, B x–xii], Immanuel Kant saw that the gap must have been bridged, perhaps by Thales (who, according to Herodotus [17, I.74], predicted the year of a solar eclipse, which must have been that of 585 B.C.E., almost 300 years before the flourishing of Euclid):

> A new light broke upon the person who demonstrated the isosceles triangle (whether he was called "Thales" or had some other name). For he found that what he had to do was not to trace what he saw in this figure, or even trace its mere concept, and read off, as it were, from the properties of the figure; but rather that he had to produce the latter from what he himself thought into the object and presented (through construction) according to *a priori* concepts, and that in order to know something securely *a priori* he had to ascribe to the thing nothing except what followed necessarily from what he himself had put into it in accordance with its concept.

According to the editors and translators of the *Critique* in the Cambridge Edition, Kant wrote elsewhere that by "the isosceles triangle," he meant the proposition—which became I.5 in the *Elements*—whereby the base angles of an isosceles triangle are equal to one another.

A carpenter or surveyor will recognize that many of Euclid's propositions *are* true of the world. Euclid shows why they *must* logically be true, on the basis of some plausible hypotheses. This very plausibility may militate against the full understanding of mathematics as being hypothetical.

In their first semester in our department, mathematics students at Mimar Sinan present to one another the propositions of the first book of Euclid's *Elements.* This is in the style of my own *alma mater,* St John's College [30]. However, at St John's, all courses require active participation in the reading of original texts, and students come to the college expecting such courses; at Mimar Sinan, they may not expect such courses, and Euclid may be the only one they get. Some of our students are excited to read Euclid, while others just do what they need to get by.

In practice, few of our students may get over the notion that geometry is a practical affair, like surveying. Surveying is the original sense of the Greek γεωμετρίη, whose origins are traced by Herodotus [17, II.109] to the need in Egypt to redraw boundaries after the annual flooding of the Nile.

In the semester after reading Euclid, our students have a conventional course of analytic geometry. Such a course relies on what in some countries, including Turkey, is called Thales's Theorem: a straight line cutting two sides of a triangle cuts them proportionally if and only if the cutting line is parallel to the third side of the triangle. I have examined this theorem at length elsewhere [31], developing it from Book I of Euclid's *Elements.* I tried doing the same thing in my department's analytic geometry course, but without much success. The intuition for Thales's Theorem developed in high school is normally too strong to be questioned.

A course of non-Euclidean geometry will bring one's intuition into question. I have taught such a course as an elective twice so far, once at Mimar Sinan, once at the Nesin Mathematics Village. We started with a study of Pappus's Theorem, as presented by Pappus himself [26, 27, 28], for the sake of developing projective geometry; we continued with Lobachevski [24], for hyperbolic geometry. I am not sure how many students could get over their conviction that space was simply Euclidean.

In set theory, with the study of the ordinal numbers, one's intuition for the real numbers ought to be an advantage, even as, with some new hypotheses, one goes on to create a new world.

3 Hypotheses

3.1 Euclid and Archimedes

At the beginning of Heiberg's edition of the *Elements* [12], some hypotheses are written out in the form of *postulates* (*αἰτήματα* "requests") and *common notions* (κοιναὶ ἔννοιαι). There are five of each kind. However, the extant manuscripts vary, and there is still some question of what was actually written by the original compiler of the *Elements,* the compiler whom we call Euclid.

By the common notions, equality is what is now called, in universal algebra, a *congruence* with respect to addition. Equality is also congruence in the original sense of being able to coincide: where Heath's English [13] for the fifth common notion is, "Things congruent to one another are equal to one another," Heiberg's Latin [12] reads, "QUAE INTER SE CONGRUUNT, AEQUALIA SUNT" (see [32, §2.4]).

I read Euclid's first four postulates as hypothesizing the availability of a minimal toolbox. By the first three postulates, this toolbox contains a straightedge (or perhaps a chalk line) for drawing and extending straight lines, and a compass (or again a *line,* in the sense of a string or cord) for drawing circles. The fourth postulate provides a set square, not for drawing right angles, but as a symbol of the hypothesis that all right angles, once drawn, are equal to one another. The fifth postulate ensures that a point of intersection of two straight lines can be found, provided those straight lines are crossed by another, which makes interior angles on the same side together less than two right angles.

A. Seidenberg is reluctant to treat all of Euclid's postulates as axioms in the modern sense [40, p. 294]:

> The construction postulates are *bona fide* and are axioms in a sense: they serve to control the straightedge and compass constructions. But they are not axioms for a development of geometry and, indeed, tell us nothing about space, except incidentally that there is a line on any two points. As to Postulates 4 and 5, there is the tendency to attribute profound significance to them: and whoever made Postulate 5 into an explicit assumption, if he did not do it as a simple comment, deserves great credit; but Postulate 4 is not needed, as already observed by the ancients, and its inclusion as an assumption is inept.

We may thus indeed contrast Euclid's explicit hypotheses with the kind of hypotheses that we make today, or even that Archimedes made. Euclid cannot prove that the circumferences of circles are to one another as the diameters, since he has no way to compare curved lines. Archimedes has a way, and this is to postulate, in Netz's faithful translation [1, p. 36],

> /1/ That among lines which have the same limits, the straight <line> is the smallest. /2/ And, among the other

> lines (if, being in a plane, they have the same limits): that such <lines> are unequal, when they are both concave in the same direction and either one of them is wholly contained by the other and by the straight <line> having the same limits as itself, or some is contained, and some it has <as> common; and the contained is smaller.

In Seidenberg's view [40, p. 283]:

> EUCLID would have been thunderstruck! It would never have occurred to him that to prove a theorem ("the arc is greater than the chord"), it is all right to generalize it, and then assume the generalization.

Even so, Euclid's laying out of his postulates seems to be an advance on the hypothetical reasoning in mathematics that Plato considered earlier.

3.2 Socrates and Plato

Euclid is thought to have flourished a century after the Athenian democracy condemned Socrates to death in 399 B.C.E. For the character of Socrates in Plato's *Republic,* mathematics does not represent the highest truth, precisely because it is based on hypotheses; but Socrates seems to have a less developed notion of hypothesis than Euclid's.

Even what Socrates says about mathematics in the *Republic* may well be more than what the actual Socrates said. The *Republic* distinguishes mathematics and philosophy by means of the so-called Divided Line; however, this line does not arise in Socrates's "intellectual biography" in the *Phaedo.* Collingwood suggests therefore, in his 1933 *Essay on Philosophical Method* [8, p. 12], that the Divided Line must be Plato's own innovation.

Socrates himself may not have distinguished between mathematics and philosophy. For him, these were both to be pursued by *dialectic,* or the technique of question and answer [8, p. 11]:

> This technique, as he himself recognized, depended on a principle which is of great importance to any theory of philosophical method: the principle that in a philosophical inquiry what we are trying to do is not to discover something of which until now we have been ignorant, but to know better something which in some sense we knew already; not to know it better in the sense of coming to know more about it, but to know it better in the sense of coming to know it in a different and better way . . .

Collingwood here could be describing Euclid. In the *Elements,* Euclid may not give the world new mathematical *facts*; but he gives us a new and better way of thinking about them.

In Book VI of the *Republic,* Glaucon summarizes an argument of Socrates [33, 511D]:

> I think you call the mental habit of geometers and their like *mind or understanding* (διάνοια) and not *reason* (νοῦς) because you regard understanding as something intermediate between *opinion* (δόξα) and reason.

Socrates replies, "Your interpretation is quite sufficient."

I have given here, and shall continue to give, the original Greek of some key terms. These and their variants have often become technical terms in English; but we should not consider them as technical terms in Greek. In particular, they will not have precise definitions. We have just seen a description of three forms of thought. However, the four sections of the Divided Line correspond to *four* modes of thought. In the account of Glaucon, there may be something below opinion, or else opinion itself should be of two kinds, referred to respectively as *belief* (πίστις) and *picture-thinking or conjecture* (εἰκασία) [33, 511E].

The Divided Line is first divided in an unequal ratio, and Socrates describes the two sections as corresponding respectively to what is merely *visible* (ὁρατός) and what is *intelligible* (νοητός) [33, 509D]. Each of the two sections is in turn divided in the same ratio. The corresponding division of the visible is into images, such as shadows or reflections, and those things that actually cast the shadows or are reflected. The division of the intelligible is made [33, 510B]

> by the distinction that there is one section of it which the *soul* (ψυχή) is compelled to investigate by treating as images the things imitated in the former division, and by means of *assumptions* (ὑποθέσεις) from which it proceeds not up to a *first principle* (ἀρχή) but down to a *conclusion* (τελευτής), while there is another section in which it advances from its assumption to a *beginning or principle that transcends assumption* (ἀρχή ἀνυπόθετος), and in which it makes no use of the *images* (εἰκόνες) employed by the other section, relying on *ideas* (εἴδη) only and progressing systematically through ideas.

To clarify the meaning of hypothetical thinking, of reasoning from assumptions to conclusions, Socrates describes mathematics [33, 510C]:

> students of *geometry* (γεωμετρία) and *reckoning* (λογισμός) and such subjects first *postulate* (ὑποθέμενοι) the odd and the even and the various *figures* (σχήματα) and *three kinds of angles* (γωνιῶν τριττὰ εἴδη) and other things akin to these in each *branch of science* (μέθοδος), regard them as known, and, treating them as *absolute assumptions* (ὑποθέσεις αὐτά),[1] do

[1]Plato's habit, shown here, of attaching a neuter form of the adjective αὐτός to any noun, regardless of gender, is recognized by the *Greek–*

> not deign to render any further *account* (λόγος) of them to themselves or others, *taking it for granted* (ἀξιοῦσι διδόναι) that they are obvious to everybody.

The hypotheses discussed here may resemble Euclid's first three postulates, though they are rather different from his fourth and fifth postulates. The latter kind of hypothesis concerns what we can *infer* from an application of the former kind of hypothesis.

3.3 Metaphysics

In his 1940 *Essay on Metaphysics,* Collingwood argues that the meaning of hypothesis in the *Republic* has been misunderstood, particularly at the point when Socrates asks [33, 533C],

> is not *dialectics* (ἡ διαλεκτική) the only *process of inquiry* (μέθοδος) that advances in this manner, *doing away with hypotheses* (ὑποθέσεις ἀναιροῦσα), up to the *first principle* (ἀρχή) itself in order to find confirmation there?

In the chapter of Collingwood's *Essay* called "A Positivistic Misinterpretation of Plato" [7, ch. XV], the misinterpretation referred to by the title is that "doing away with hypotheses" means doing away only with their hypothetical character, but establishing them instead as truths of experience. The structure of the *Republic* itself shows that this not what Socrates means. The dialogue begins with the hypothesis, accepted tacitly by Polemarchus, that justice is a form of art or skill (τέχνη). Through dialogue, this hypothesis is done away with by being shown untenable; so is Thrasymachus's hypothesis that *in*justice is a skill.

English Lexicon of Liddell and Scott [23].

Thus argues Collingwood, plausibly to my mind. Speaking as an *amateur* of philosophy, I find it tragic that Collingwood is little read today. His work shows the foolishness of some common thoughts today: that physics can come up with a "theory of everything," and neuroscience can explain consciousness. A proper theory of *everything* would include, for example, a theory of art; the theory would not "reduce" art to something it was not. Influenced by Collingwood's 1938 *Principles of Art* [6], I consider art as a process of bringing emotions to consciousness. Whatever it is, art aims to *do* something, and to do it *successfully.* Art can be good or bad; but the distinction between good and bad is not one that is made by a natural science.

Any science aims to be true, and this means it has standards for what are good examples of its work. These standards may be maintained by such conventions as peer review of articles. But the study and application of the standards is not normally a part of the science itself. The best mathematician is not expected to be either the best expositor of mathematics or the best referee of a mathematical article.

Physics and engineering can send a probe through the rings of Saturn, but they do not decide whether such a probe ought to be sent in the first place. *Physicists and engineers* may decide, in their broader capacity has human beings. Nobody is *simply* a scientist, a mathematician, or an artist; nor, perhaps, should anybody try to be.

There *are* sciences that include themselves among their objects of study. These sciences are what Collingwood calls *criteriological* [7, p. 109], because they are concerned with the criteria whereby a piece of work is judged successful *by the entity that is trying to do the work.* Logic and ethics are criteriological sciences. An essay on logic aims to *be logical,* by its own standards; and essay on ethics should be ethical (as by citing sources, not plagiarizing, and generally not lying). A logician or ethicist may fail to be logical or ethical; but if they are not even *trying* to be logical or ethical, then, strictly speaking, they are not being a logician or ethicist: either they do not know what logic and ethics mean, or else they have forgotten.

Except perhaps by astrologers, Saturn is not considered to be *trying* to do anything: it is just *there.* Thus the way Saturn is studied is different from the way an article about Saturn is studied. The reader of the article will ask not what Saturn is trying to do, but what the article is trying to say.

Therefore I think Edward O. Wilson is mistaken when writing, in a September 2014 article in *Harper's* magazine [47],

> Philosophers have labored for more than two thousand years to explain consciousness. Innocent of biology, however, they have for the most part gotten nowhere. I don't believe it too harsh to say that the history of philosophy when boiled down consists mainly of failed models of the brain.

What Wilson says is not harsh, but wrong, and foolishly so. The Divided Line in the *Republic* is not a model of the brain; it is a model of thinking, a model that recognizes a *better* and *worse* in thinking. As an organ studied by biology, the brain does not judge itself, any more than the heart judges whether it is pumping well. The heart does what it does; the physician decides whether what the heart does is good. Thought asks itself whether it is proceeding well; but this is not a biological question.

For the teacher of mathematics, the Divided Line is a good model of thinking, since it distinguishes opinion from reason or knowledge, and it distinguishes hypothetical thinking from another kind—namely categorical thinking, in the terminology of Collingwood's *Essay on Philosophical Method* [8, p. 121]. Because a teacher told them so, students may be of the *opinion* that the quadratic equation

$$ax^2 + bx + c = 0$$

has two solutions, which are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The serious teacher wants more. Students should *know* that the quadratic equation is thus solved, at least because they can check the given solution by substitution, but preferably because they can derive the solution by completing the square. Ultimately they should also know that the solution depends on the hypothesis of working over the complex numbers, or perhaps an arbitrary quadratically closed field; they may come to know this by seeing that, in the ring of integers *modulo* 8, the quadratic equation $x^2 - 1 = 0$ has four solutions.

In *An Essay on Metaphysics,* in his discussion of hypotheses in the *Republic,* Collingwood says [7, p. 156–7],

> In mathematics—I take my example from the kind of mathematics which people were supposed to know by the time they began studying philosophy under Plato—you begin a job of thinking by doing something that is enjoined in the words 'Let ABC be a triangle, and let the angle ABC be a right angle'. Then you try to show that the square on AC is equal to the sum of the squares on AB and BC. What you do at the start, what you were told to do in the words 'Let ABC', &c., is making, or positing, or setting up, a supposition which relatively to the rest of your thinking in this particular job is a presupposition or 'hypothesis'.

Collingwood here describes roughly how one sets out to understand the next-to-last proposition in Book I of Euclid's *Elements.* In thus setting out, one not only posits a right triangle, but one allows oneself to use the tools of straightedge, compass, and set square in the ways prescribed by the postulates. Plato seems not to have recognized the possibility of such postulates as hypotheses. Euclid recognized them, but now Collingwood seems to have overlooked them.

The kind of hypothesis represented by Euclid's postulates is nonetheless essential to mathematics. The mere fact of using hypotheses does not distinguish mathematics from natural science, since as Collingwood observes in *An Essay on Philosophical Method* [8, p. 120],

> the universal propositions laid down by empirical science have a hypothetical character not unlike that of mathematical propositions. The statement in a medical or botanical text-book that all cases of tuberculosis or all rosaceae have these and these characteristics, turns out to mean that the standard case has them; but it does not follow that the standard case exists; it may be a mere *ens rationis.*

If the botanical text-book makes a universal statement about members of the rose family, I suppose the intention is that the budding botanist will be able to use this statement during any future encounter with a flower that might be a rose.

Likewise, the calculus student who learns the Intermediate Value Theorem should be able to apply it to any continuous function. However, functions are not quite like flowers. The calculus student will also learn the theorem that all polynomial functions are continuous, so that every polynomial of odd degree has a zero; but here the hypothesis of working over the real numbers must not be forgotten, lest the student think that the polynomial $x^3 + 2$ has a root, even in the ring of integers *modulo* 8.

I do not actually recall a student's making this confusion. I do recall being a teaching assistant for a linear algebra lecture, and one of my fellow assistants was disturbed that the undergraduates were being told falsehoods, such as that not every square matrix had an eigenvalue. We were working over the real numbers; but my fellow graduate student evidently thought every matrix should "really" be understood over the complex numbers. This may have been true for her; but she was studying so-called *applied* mathematics, which might indeed be understood as a natural science.

3.4 Accounting

Regardless of any technical differences in their mathematics, Euclid has the spirit of Socrates, who at his trial recalls the pronouncement of the oracle at Delphi that he, Socrates, was the wisest of men [34]. Incredulous, Socrates investigated the men who were reputed to be wise. They could not give an account, even of what they were supposed to be wise about:

> After the political experts I went on to the poets . . . on the basis that it was here that I'd catch myself red-handed, as actually more ignorant then them. So, picking out those of their poetic compositions they seemed to me to have spent most effort on, I would ask them what they were trying to say, with a view to learning a thing or two from them as well. Well, Athenians, I blush to tell you the truth, but it has to be told: practically speaking, almost everyone present would have better things to say than they did about their own compositions . . . But, men of Athens, the good craftsmen too seemed to me to suffer from the same failing as the poets: because they were accomplished in practising their skill, each one of them claimed to be wisest about other things too, the

> most important ones at that—and this error of theirs seemed to me to obscure the wisdom that they did possess.

As Socrates investigated what was taken for wisdom in politics, poetry, and craft, so Euclid investigates what is taken for wisdom in mathematics.

Most undergraduate mathematics has an obvious meaning in the physical world, or at least is of use to experts who work in the world. Even number theory has its applications to cryptography, and thus to warfare and internet commerce. But mathematics proper must be able to explain not only how it is true, but why. *How* Newton's infinitesimal calculus is true is that it can successfully derive the observed motions of the planets from an inverse-square law of gravitation [10, 25]. But this success does not quite make calculus into mathematics. *Infinitesimal* calculus became mathematics as such only after three centuries, when Abraham Robinson founded it in logic [37]. Euclid had shown the way to do this, or at least he had established the ideal, more than two millenia earlier.

Set theory may now take the place of number theory as the purest mathematics. But it was created to explain the power of calculus, before Robinson was born. This power comes precisely from the use of the infinite and infinitesimal. Alexandre Borovik [3, p. 111] notes the paradox (attributed to Arnol′d) that infinity is useful as an approximation to the very large, even though there may be nothing infinite in the world.

The linguist says we have the capacity to form an infinite number of grammatical sentences, just as, in symbolic logic, by varying the counter n, we can create infinitely many conjunctions of the form

$$P_1 \wedge \cdots \wedge P_n.$$

The infinite number of such conjunctions is still only a so-called *countable* number. However, even the *uncountably* infinite is useful, in the form of the real numbers themselves, which compose an uncountably infinite set. Understanding this is another basic application of set theory.

4 Standards

4.1 Personal

Taking inspiration from the line of real numbers, I try to develop (in my set-theory course) an analogous conception of the class of all ordinal numbers. Treating an ordered pair (ξ, η) of ordinals as if it were an ordered pair (x, y) of real numbers, I draw graphs of continuous and non-continuous ordinal-valued functions, as in Figures 1 and 2.

Most students come to us without knowing what mathematics is. For them, it is just like physics or any other course they have to take. Their job is to satisfy their teachers. *Our* job is to induce the students to insist on satisfying themselves. What we ask them to learn should be justified to *their* standards. Therefore they must develop their own standards.

This is true in any course of study; but in mathematics, at least, the standards of truth are universal. We expect universal agreement on whether a given theorem follows from given axioms and definitions. In practice, there may not be agreement, but in this case we have a clear procedure for resolving the dispute. The person who says that a theorem is true must be prepared to supply a proof, explained as needed, to anybody who is seriously interested. Teachers may demand this of students; students must also feel free to demand this of

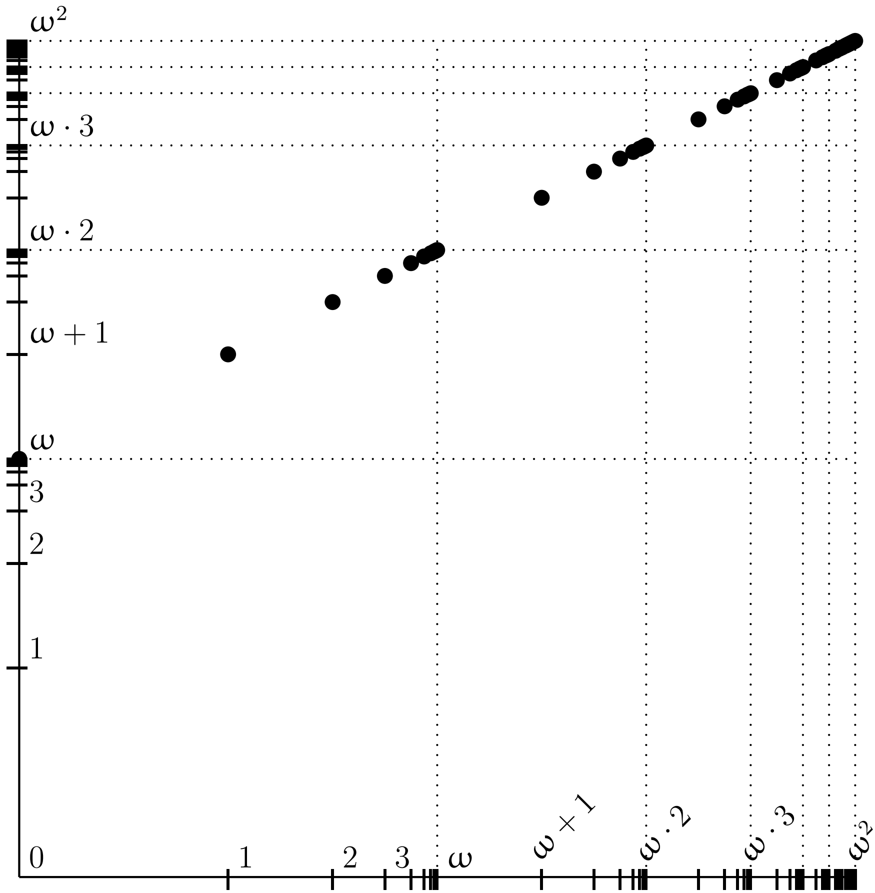

Figure 1: The graph of $\eta = \omega + \xi$

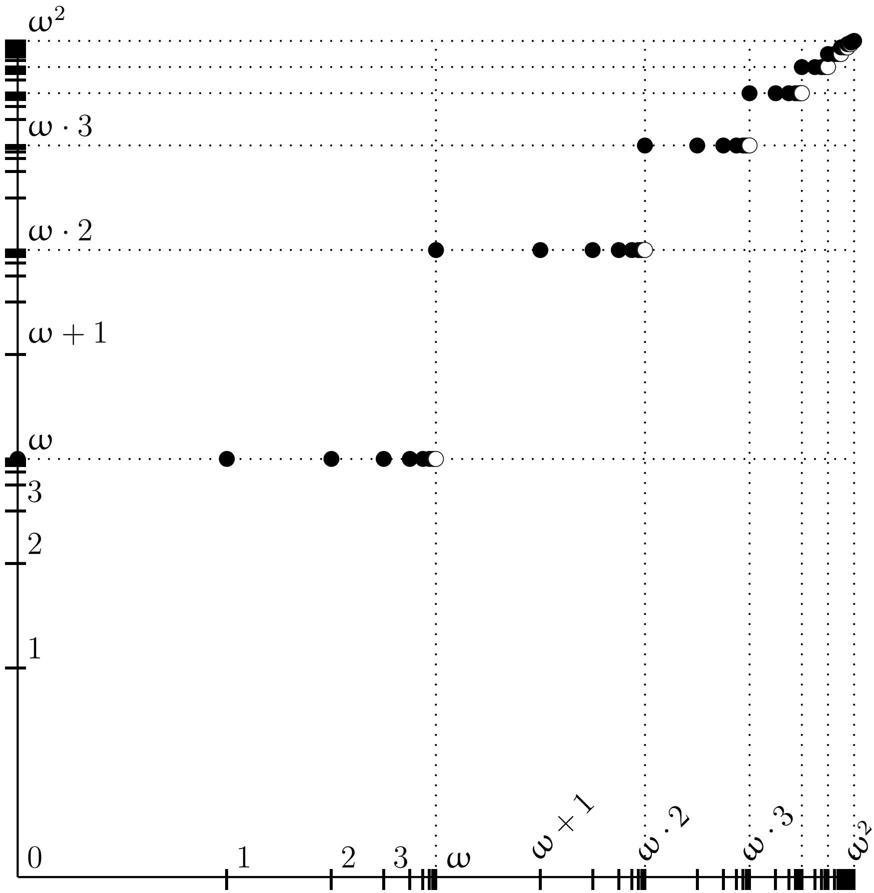

Figure 2: The graph of $\eta = \xi + \omega$

teachers.

I would echo the words of Henry David Thoreau in the Conclusion of *Walden* [44, pp. 322–3]. Thoreau first recalls an early leader of the French Revolution, who romanticized illegal violence as requiring the greatest resolution and courage. Thoreau observes that Mirabeau missed the point:

> A saner man would have found himself often enough "in formal opposition" to what are deemed "the most sacred laws of society," through obedience to yet more sacred laws, and so have tested his resolution without going out of his way. It is not for a man to put himself in such an attitude to society, but to maintain himself in whatever attitude he find himself through obedience to the laws of his being, which will never be one of opposition to a just government, if he should chance to meet with such.
>
> I left the woods for as good a reason as I went there. Perhaps it seemed to me that I had several more lives to live, and could not spare any more time for that one . . .

Live by your own laws. This is actually a requirement of mathematics. A theorem is not a theorem unless you freely agree that it is, out of your own reasoned conviction.

4.2 Universal

It is possible to be a crank, convinced of your own truth, regardless of what everybody else thinks. Lacking universality, your truth cannot be mathematical.

To earn money for going to live in the woods in the first place, Thoreau had engaged in some practical mathematics: "surveying, carpentry, and day-labor of various other kinds in the village" [44, p. 58]. He had been an undergraduate at Harvard, or Cambridge College as apparently it was known then. I don't think Thoreau would claim to have learned the "laws of his being" at Harvard. But the experience must have helped him to become himself.

Heading out to catch fish for his supper one day, Thoreau was caught by the rain, and he sought shelter in a hut that he thought empty. It turned out to be in use by an immigrant family from Ireland. "An honest, hard-working, but shiftless man plainly was John Field" [44, p. 204]; but the young Henry David thought he could give John and his wife some advice on how to live better.

> I tried to help him with my experience, telling him that he was one of my nearest neighbors, and that I too, who came a-fishing here, and looked like a loafer, was getting my living like himself; that I lived in a tight, light, and clean house, which hardly cost more than the annual rent of such a ruin as his commonly amounts to; and how, if he chose, he might in a month or two build himself a palace of his own; that I did not use tea, nor coffee, nor butter, nor milk, nor fresh meat, and so did not have to work to get them; again, as I did not work hard, I did not have to eat hard, and it cost me but a trifle for my food; but as he began with tea, and coffee, and butter, and milk, and beef, he had to work hard to pay for them, and when he had worked hard he had to eat hard again to repair the waste of his system—and so it was as broad as it was long, indeed it was broader than it was long, for he was discontented and wasted his life into the bargain; and yet he had rated it as a gain in coming to America, that here you could get tea, and coffee, and meat every day.

It may have been only by being educated that Thoreau could question the value that society placed on luxuries of the flesh, and could see that

> the only true America is that country where you are at lib-

> erty to pursue such a mode of life as may enable you to do without these, and where the state does not endeavor to compel you to sustain the slavery and war and other superfluous expenses which directly or indirectly result from the use of such things.

Thoreau does presently allude to his education. He is aware of society's scorn for somebody who does not use an education for the practical purpose that it was supposedly intended for:

> As I was leaving the Irishman's roof after the rain, bending my steps again to the pond, my haste to catch pickerel, wading in retired meadows, in sloughs and bog-holes, in forlorn and savage places, appeared for an instant trivial to me who had been sent to school and college; but as I ran down the hill toward the reddening west, with the rainbow over my shoulder, and some faint tinkling sounds borne to my ear through the cleansed air, from I know not what quarter, my Good Genius seemed to say—Go fish and hunt far and wide day by day—farther and wider—and rest thee by many brooks and hearth-sides without misgiving. Remember thy Creator in the days of thy youth. Rise free from care before the dawn, and seek adventures.

Education *is* an adventure, whose purpose is to make it possible to seek more adventures. Mathematics is an adventure, set theory is an adventure. Thoreau said that if you followed the laws of your own being, you would break no laws of any just government. This kind of teaching can be abused; but it is fundamental to mathematics. You have it in yourself to decide what is true; or if you haven't, then nobody else can decide for you. And yet your decision will be in harmony with that of everybody else, or at least of everybody else who cares to make a decision.

This does not mean that everybody's mathematics is the same. I once had a student in a set-theory course who thought a certain equation was true because he had seen it in an authoritative source. He had not understood that, in his source, the equation was true by a definition that my own course had not adopted. The equation was

$$\bigcap \varnothing = \varnothing. \tag{1}$$

I had taught instead

$$\bigcap \varnothing = \mathbf{V}, \tag{2}$$

$\mathbf{V}$ being the universal class $\{x\colon x = x\}$; but equation (2) was for me a *theorem,* which followed from the natural definition

$$\bigcap \boldsymbol{C} = \{x\colon \forall Y\ (Y \in \boldsymbol{C} \Rightarrow x \in Y)\},$$

where $\boldsymbol{C}$ is any class. This is what I have always taught. With this definition in mind, one can write a "De Morgan law" in the form

$$\Bigl(\bigcap \boldsymbol{C}\Bigr)^{\mathrm{c}} = \bigcup \{X^{\mathrm{c}}\colon X \in \boldsymbol{C}\}, \tag{3}$$

which is a generalization of the more familiar

$$(A_1 \cap \dots \cap A_n)^{\mathrm{c}} = {A_1}^{\mathrm{c}} \cup \dots \cup {A_n}^{\mathrm{c}}. \tag{4}$$

One has to explain the notation on the right of (3), since the complement of a set is a proper class, and proper classes are not elements of classes, so $\{X^{\mathrm{c}}\colon X \in \boldsymbol{C}\}$ is not a class unless $\boldsymbol{C} = \varnothing$. One can still define

$$\bigcup \{X^{\mathrm{c}}\colon X \in \boldsymbol{C}\} = \{x\colon \exists Y\ (Y \in \boldsymbol{C} \land x \notin Y)\}.$$

When $\boldsymbol{C}$ is empty, one gets

$$\bigcup \varnothing = \varnothing, \tag{5}$$

and so (2) follows by use of (3). Also (2), or rather

$$\left(\bigcap \varnothing\right)^{\mathrm{c}} = \varnothing,$$

is the natural interpretation of (4) when $n = 0$.

One could say that (4) made no sense when $n = 0$; but this would be contrary to the spirit of generalization in mathematics: a spirit which may lead to vacuity, but also to insight and simplicity. In *Mathematics: A Very Short Introduction* [16, ch. 3, pp. 35–48], Gowers notes that the straight lines joining any two of n points on a circle divide the circle into 2^{n-1} regions when $1 \leqslant n \leqslant 5$. This is not a proof that the same is true for all n:

> In fact, with a little further reflection one can see that the number of regions *could not possibly* double every time. For a start, it is worrying that the number of regions defined when there are 0 points round the boundary is 1 rather than 1/2, which is what it would have to be if it doubled when the first point was put in. Though anomalies of this kind sometimes happen with zero, most mathematicians would find this particular one troubling.

That a certain formula does not work when $n = 0$ is a sign (though not a conclusive one) that the formula will not work for other n either.

It is clear what the sum $\sum_{i=1}^{n} a_i$ means when n is a counting number; and when $n = 0$, the sum should be zero. But then the product $\prod_{i=1}^{n} a_i$ should be 1 when $n = 0$, since 1 is neutral with respect to multiplication. We define

$$n! = \prod_{i=1}^{n} i, \qquad\qquad 0! = 1.$$

Likewise, since the universal class $\mathbf{V}$ is the neutral class with respect to intersection, we should define (2) to be true.

Nonetheless, the elegance of (2) may be lost on students, and even on some mathematicians. When the student whom I mentioned earned no credit on an examination for writing (1) instead of (2), he showed me an issue of of the Turkish magazine *Matematik Dünyası* [Mathematics World] in which Ali Nesin said that (1) was correct. In his own development of set theory, Nesin worked only with sets, not with proper classes. In particular, he defined intersections only of sets, and these intersections should always be sets. By *fiat* then, (1) held, there being no better alternative. Seeing this equation, apparently my student took it to be as true as an equation from physics like $\mathbf{F} = m\mathbf{a}$. He had not learned that Arnol′d was not quite right to say what I quoted earlier, about how mathematics was a part of physics. The very "cheapness" of its experiments makes mathematics different. In every class, using axioms and definitions, we can create a new world, which need not be the same as the world seen in a previous class, or in a text that we are not using.

I cannot find it!

As the student of linear algebra must learn to work with more than three dimensions, overcoming his preconception that the additional dimensions have no physical meaning, so the student of set theory must learn to work with infinite sets that are strictly larger than other infinite sets, and even with classes that are too large to be sets at all. The universal class $\mathbf{V}$ might be considered as analogous to the point ∞ at infinity, which the student has already seen in calculus.

5 The writing of textbooks

I have now taught set theory three times at Middle East Technical University in Ankara, and three times at Mimar Sinan. I have always produced my own text for the course. The text has normally contained more than can be covered in the course, because I have written the text more for the teacher (namely myself) than for the students. Most textbooks may be written so that they can replace a teacher's lectures. I myself have aimed to put in my own texts everything that lectures will cover; but the text may be terser than the lectures. The text also covers more: things that *I,* at least, want to know, or that that are needed to satisfy some notion of formal completeness, though they can be skipped in class.

I write texts for many of my courses. For this habit, I blame the man who taught me precalculus and calculus in the last two years of high school. Donald Brown had us buy two textbooks: Spivak [41] for theory, and Salas and Hille [39] for practice. The real text for Mr Brown's course was the one that we copied down from what Mr Brown wrote on the blackboards. I learned in this way that different choices could be made about how to do mathematics. Mr Brown was making his own choices. If I became a teacher, I could make my own choices.

In a number-theory course that I taught at METU, a student complained that the text was difficult to understand. He was embarrassed when I pointed out that the text was by me.[2] I told him that the real course was worked out in the classroom. If a student could follow the text alone, that was fine; but I was not interested in producing a text that would obviate a student's need to come to class.

I may not be able to produce such a text. Perhaps few per-

[2] I met him in May of 2016, and he was embarrassed that I still remembered him for his old complaint. He was still in the same department, now working on a doctorate.

sons can, but if others try, they end up with a bloated textbook with something for everybody, but too much for anybody. At least one student did praise one edition of my own set-theory text; but she had been by far the best student in the class.

My new proposed text is somewhat different from the earlier versions. It is set theory, now stripped down to what I think *all* students can be asked to learn. I have also changed my mind about what needs to be in the course.

I began teaching set theory at METU in order to work out some of my own concerns about mathematical rigor. For example, I had noticed that the logical distinction between induction and recursion in the natural numbers was not commonly recognized. Even otherwise-rigorous textbooks treated induction, strong induction, and well-ordering as equivalent principles, from any one of which, all of the other properties of the natural numbers could be derived. Mr Brown did this, as did Spivak; but they were in error. I ultimately wrote an article [29] about this. Meanwhile, I wanted to clarify matters in my own set-theory course.

I learned that my concerns were lost on most students. Since my students had entered university on the basis of their ability to solve computational problems, I decided that, in set theory, they should at least be able to perform computations with what Cantor [5, §19, pp. 183–195] calls the normal forms of ordinal numbers.

6 A set-theory text

The chapters of my text are:

1. Real Analysis
2. Ordinal Numbers

3. Set Axioms
4. Ordinal Addition
5. Ordinal Multiplication
6. Ordinal Exponentiation
7. Cardinal Powers

A. Letters
B. Logic
C. Cofinality

I describe these chapters now.

I used to want my set-theory students to learn something of symbolic logic. Now I have all but dropped this topic from the course. Students should still learn that sets can be collected into classes, which are defined by formulas; but the formal definition of formulas is now relegated to **Appendix B.**

I used to try to motivate set theory by the paradox that our earliest mathematical activity is based on a proposition that everybody accepts without proof, but that is not properly an axiom. This proposition is that no matter what order we count a set in, we always get the same number. The proposition fails when the set is infinite. This failure is a sign that the proposition for finite sets is a real theorem, not an axiom. One might alternatively conclude that there are no infinite sets, or that there is not really any way to count them. I had a friend who did not believe in infinite sets; he could have been a mathematician, but became a lawyer instead.

In earlier set-theory classes, I did postpone the Axiom of Infinity as long as possible. I did not do this in my most recent class, because I had decided to try to motivate set theory differently this time.

One does real analysis on the basis of the axioms of a complete ordered field. One can in fact construct a complete ordered field from the natural numbers, as given by the Peano

Axioms. The way was shown by Dedekind [9], and Landau [20] works out the details; but one need not construct the real numbers, in order to do analysis.

I review the approaches to real analysis in **Chapter 1,** which I discuss in more detail below. I go on to apply the possibilities to set theory. The ordinals are analogous to the real numbers, not only by being linearly ordered, but by being complete in their ordering: every nonempty set with an upper bound has a least upper bound.

In order to do "ordinal analysis" as soon as possible, I present axioms for ordinals in **Chapter 2,** and I prove from these axioms the tools needed for ordinal arithmetic: ordinal induction and ordinal recursion. Alternatively, one might just accept ordinal induction and recursion as axioms themselves, or as grand theorems whose proofs are deferred, like the Intermediate Value Theorem in some calculus books. In Salas and Hille, the proof of the IVT is in an appendix. In my own set-theory classes, I may prove the theorem of ordinary recursion (recursion on $\mathbb{N}$ or ω), to give a taste of what is involved, while waving my hands over ordinal recursion.

In just a few pages, Suppes [42, pp. 195–205] gives three versions of transfinite induction and five versions of transfinite recursion. I give only one version of each, a version obtained from ordinary induction or recursion by adding a limit step.

In **Chapter 3** are presented those set axioms with which the existence of a model of the ordinal axioms can be established. The chapter has two sections. As presented in the previous chapter, ordinal recursion gives us only functions from **ON** into itself. With the Empty-Set, Adjunction, and Replacement axioms, we can recursively define an isomorphism between any structure satisfying the ordinal axioms and the particular example defined by von Neumann [46]. We may henceforth assume that this example is **ON**. This gives us the convenient identities

$$\alpha = \{\xi \colon \xi < \alpha\}, \qquad\qquad \sup A = \bigcup A,$$

for elements α and subsets A of **ON**. These identities are found in the first section of the chapter. In the next section, we observe that the new **ON** is transitive and well-ordered by $\in$, and so are all of its elements. This lets us define **ON** without using the ordinal axioms.

Bruno Poizat might be critical here. In his *Cours de Théorie des Modèles* [35, §8.1, p. 213], translated as *Course in Model Theory* [36, p. 163], after showing that every well-ordered set is isomorphic to a unique von Neumann ordinal, he writes,

> *On rencontre des étudiants qui sont allergiques aux ordinaux comme "type de bons ordres", et qui trouvent plus digeste la notion d'ordinal de Von Neumann ; c'est là une singulière conséquence d'un enseignement dogmatique, qui confond formalisme et rigueur, et qui met en avant l'astuce technique au détriment de l'idée fondamentale : il faut un esprit étrangement déformé pour trouver naturelle la notion d'ensemble transitif !*
>
> We meet some students who are allergic to ordinals as "well-ordering types" and who find the notion of von Neumann ordinals easier to digest; that is a singular consequence of dogmatic teaching, which confuses formalism with rigor, and which favors technical craft to the detriment of the fundamental idea: It takes a strangely warped mind to find the notion of a transitive set natural!

Perhaps "craftiness" or "trickery" would be a better translation than "craft" for Poizat's *astuce.* In any case, I would not expect the mathematician to be the best judge of what is "strangely warped" (*étrangement déformé*). Different people think differently, even within mathematics. Our students are indeed victims of dogmatic teaching; but it is they with whom we have to work.

In real analysis, each real number can be understood as the set of all rational numbers that are less than itself; but one need not have this understanding, in order to *do* real analysis. The understanding may even be a distraction. In ordinal analysis, we need only think of ordinals as points on a certain line. However, it *is* useful for our purposes if the ordering of that line is set-theoretic membership, so that every ordinal is precisely the set of ordinals that are less than itself. For one thing, this means an ordinal has an intrinsic cardinality. In an early draft of my text, I had written a theorem in the form,

$$\alpha \cdot \beta \approx \{\xi \colon \xi < \alpha\} \times \{\xi \colon \xi < \beta\}.$$

At some stage, this may express the theorem better than what I have now written, namely

$$\alpha \cdot \beta \approx \alpha \times \beta.$$

But it seemed to me that maintaining a formal distinction between α and $\{\xi \colon \xi < \alpha\}$ would be perverse. At least it would be verbose.

Adjunction is not one of Zermelo's original axioms [48], but it follows from his axioms of Union and "Elementary Sets" (classes of at most two elements are sets). I prefer to give Adjunction before having to introduce Union. Once the Union Axiom *is* introduced, along with Separation and Infinity, we can show that von Neumann's ordinals do exist so as to satisfy the ordinal axioms given earlier. However, this material is independent from the rest of the text, and perhaps it can be ignored, however paradoxical that may be for a course of axiomatic set theory.

Most of the Zermelo–Fraenkel axioms can be understood to be that certain classes are sets. I have never seen the Axiom of Infinity presented this way, even in a thorough discussion of the axioms by Fraenkel and others [14]. However, since we have already introduced the ordinals, we can let the Axiom of Infinity be that the class of all ordinals that neither *are* limits nor *contain* limits is a set.

The Power-Set Axiom appears only in **Chapter 7,** on cardinals, so that we can establish that there are uncountable sets. Then the Axiom of Choice gives us that every set is equipollent with some ordinal. Defining the Beth numbers as well as the Aleph numbers makes some nontrivial computational problems possible. It may be a perversity of mathematics that we look for ways to give students problems; but this is what I have done.

I never mention the Foundation Axiom. One may raise the question of whether a set can be a member of itself; but I see no point in declaring that it cannot, unless one is going to give the proof that such a declaration is consistent with the other axioms.

It might be said that my use of $\mathbf{V}$ for the class of all sets implies my acceptance of the Foundation Axiom. I use this notation in the text only to point out that $\mathscr{P}(\mathbf{V}) = \mathbf{V}$, so that Cantor's Theorem $A \prec \mathscr{P}(A)$ must somehow use that A is not a class. I do not pause to point out that $\bigcap \varnothing = \mathbf{V}$.

The possibility of computational problems with ordinals is developed in **Chapters 4, 5,** and **6,** which concern ordinal addition, multiplication, and exponentiation respectively. The chapters are laid out in parallel, as far as possible. Thus in **Chapter 4** we establish that each element of ω^2 is of the form $\omega \cdot k + n$; and also $n + \omega = \omega$. In particular then, ω^2 is closed under addition. In **Chapter 5,** we see that ω^ω

is closed under addition and multiplication. Within this set, Cantor normal forms can be defined in close analogy with the usual positional notation for counting numbers: any element of ω^ω can be written as

$$\omega^k \cdot m_0 + \omega^{k-1} \cdot m_1 + \cdots + \omega \cdot m_{k-1} + m_k$$

for some k in ω, where the coefficients m_i are allowed to be 0. The rules for addition and multiplication in ω^ω are as challenging as those for arbitrary Cantor normal forms, and they are introduced before arbitrary exponentiation is worked out in **Chapter 6.**

The tools are all there; but I think there is no need to give a rule for raising an arbitrary Cantor normal form to the power of an arbitrary Cantor normal form.

The chapters on ordinal arithmetic establish that ordinal sums, products, and powers are respectively equipollent with disjoint unions, cartesian products, and sets of finitely supported functions. It is established within each chapter that the corresponding operation yields only countable sets when applied to countable sets.

In **Chapter 7,** the Power-Set Axiom gives us uncountable sets. That cardinal addition and multiplication are trivial is established by use of Cantor normal forms.

For completeness, I added the topic of cofinality to an earlier edition of the text. It allows precise computation of infinite cardinal powers, provided one grants the Generalized Continuum Hypothesis. I have never had time to talk about this in class; but the material is in **Appendix C.**

Chapter 1 is an attempt to introduce foundational thinking in a familiar context: the real numbers. Given the real numbers as constituting an ordered field, I define the natural numbers as certain real numbers (which is what Spivak does). Ultimately I prove the Peano Axioms as a theorem about these natural numbers. Many details are left as exercises; in the class itself, I had students present at the board their solutions (or the solutions that they had looked up).

One could however skip **Chapter 1.** As it is, the text introduces *four* lists of axioms: (1) the axioms of a complete ordered field, (2) the Peano axioms, (3) axioms for the class of ordinals, and (4) the Zermelo–Fraenkel axioms. One could skip the first two lists and treat the third explicitly as a theorem whose proof is deferred. Or one could skip all but the third list, using "naive" set theory to justify what is done with it. This would take the course as close as possible to physics, in the sense of being about something—the class of ordinals—that is as real as the line composed of real numbers.

Appendix A lays out the different kinds of letters used as symbols in the text. It might be desirable to compose mathematics in such a way that it could be written out with a standard old-fashioned typewriter. However, the present text takes advantage of the distinctions between:

- the Latin and Greek alphabets;
- the upper and lower cases;
- roman and italic "shapes";
- plain and bold "weights," along with "blackboard bold" and curly fonts;
- different intervals of an alphabet.

Letters from the beginning of an alphabet are usually constants; from the end, variables. This follows the convention going back to Descartes's *Geometry* [11] whereby, in an equation like

$$ax^2 + bx + c = 0,$$

the x is the variable or "unknown." The a, b, and c are constant for the sake of solving the equation; but they are still variable in the sense of having no fixed values outside the solution of the equation. If they did have values that were fixed throughout the text, the letters could show this by being printed upright. In this way, upright $\upomega$ is the set of natural numbers, as opposed to italic ω, which could be an arbitrary ordinal. To avoid confusion though, ω is never used in the text.

A sort of exception to the rule on letter "shape" is that, in my usage of Ralph Smith's Formal Script, from the `mathrsfs` package for LaTeX, $\mathscr{P}$ always denotes the operation of producing the power set of a set, while letters like $\mathscr{A}$ from the same font are not fixed. Moreover, while f always denotes a function, this may be a variable or constant, simply because, in my experience, there is no standard letter for a variable function. I have the sense that ordinary language uses no variables at all; only formalized language does. Thus we may say $\varphi(a)$, meaning implicitly that for all a, $\varphi(a)$ is true; or we may say $\forall x\ \varphi(x)$ with the same meaning. I do find it satisfying to use α and ξ in ordinal analysis the way we use a and x in real analysis. But making a formal distinction between constants and variables is not all that important. I do not want students to have to wonder about whether they should write α or ξ in a particular context.

Students should however learn to distinguish between sets and classes. A set may be called a, A, or $\mathscr{A}$, depending on what features are being emphasized. If it is important that the set has elements, it may be called A; if it is important that those elements themselves have elements, it may be called $\mathscr{A}$. But in fact every element of every set is a set, whose elements are therefore sets. A class that is not known to be a set will be $\boldsymbol{A}$, written with a wavy underline on the blackboard (and not with “blackboard bold”).

In the past, I used only lower-case letters for sets, so that a capital letter would always be a class. Now I have decided it is better to follow the practice of ordinary mathematics, using lower-case letters to the left of the $\in$ sign, upper-case to the right, if this is practical. It is not practical, when the letters are Greek letters standing for ordinals; but then $\in$ can be written as $<$.

7 A set-theory course

If they learn nothing else from a course of axiomatic set theory, students should learn the Russell Paradox [38]. Alternatively, they could learn just the Burali-Forti Paradox [4], which may be taken as even more integral to the course, if the course is presented as “ordinal analysis.” The paradoxes are one bit of the “mathematics as logic” that I mentioned at the beginning. Each paradox can be most succinctly expressed as the theorem that a certain class is not a set. Without classes, you have to say things like, “Not everything that you might expect to be a set can be a set.” You can say, “Not every property defines a set”; but what is a property?

Still, introducing the class as a definite concept poses difficulty. Not every writer dare be like Levy [22], who introduces classes near the beginning of his text. Levy is perverse in another way too, by formally stating the “Axiom of Comprehension”—that every formula defines a set— and then immediately proving the theorem (called “Russell’s antinomy”) that the Axiom of Comprehension is inconsistent.

I conceive of sets as already existing. Sets are *collections,* though not every collection can be expected to be a set. Here

I use the word "collection" for the most general kind of whole that has individual elements; but the Russell Paradox keeps us from defining a "most general" such thing in an absolute sense: there is no collection of all collections that do not contain themselves. In axiomatic set theory, we want to figure out *which* collections are sets, or ought to be sets. Purely for our convenience, we require every member of a set to be a set itself. One may prefer not to consider the so-called empty set as a set; but then one will have to say things like, "For all a, where a is a set or $\varnothing$." A similar problem arises in Euclid's number theory, where unity is not properly a number, but sometimes is treated as a number. In any case, since we also consider sets to be in some sense "given," we have no reason to think that any new collection of sets that we may form is already one of the given sets. This resolves the Russell Paradox.

In my 2013–4 class, I demonstrated Tarski's Undefinability Theorem [43] in the form, *Not every collection of sets is even a class.* Indeed, set theory seems to be the best context to introduce the idea of the theorem, which originates with Gödel [15]. Gödel showed how to treat every formula about (counting) numbers as a number itself, so that, given a number theory, one could write down a true statement about numbers that was not provable in the theory. It is easier to do the same thing for set theory. Given a formula φ about sets, we can consider it as a set itself, denoted by $\ulcorner\varphi\urcorner$. Then the collection of $\ulcorner\sigma\urcorner$ such that σ is a true sentence about sets is not a class. For if it were a class, defined by a formula, then some formula φ would define the class of $\ulcorner\psi\urcorner$ such that $\psi(\ulcorner\psi\urcorner)$ is false. In this case $\varphi(\ulcorner\varphi\urcorner)$ would be true if and only if it were false. All of this can be shown to interested students; but it is not in the present text.

As I mentioned, our students at Mimar Sinan have read the first book of Euclid's *Elements.* Reading this book with them caused me to recognize what they sometimes did not: equality is not identity. Euclid proves that parallelograms of the same height on the *same* base are equal, before proving that parallelograms of the same height on *equal* bases are equal. Equality here is congruence: simple congruence of line segments, and congruence of parts in the case of parallelograms.

With this example in mind, I prefer not to take equality of sets for granted, but to *define* it as having the same elements. In a word, equality of sets is sameness of extension. Then one needs the axiom that equal sets are members of the same sets. One can then prove as a theorem that equal sets are members of the same classes. This theorem is usually taken as a logical axiom, because equality is treated as identity. In this case, that sameness of extension implies equality must be taken as a set-theoretic axiom. I have swept all of this under the rug in the present text.

The approach to set theory that led to the equation (1) may not be uncommon. In this approach, everything should be a set, and its existence should be given by an explicit construction (or at least a construction that is explicitly justified by an axiom, as in the case of a power set or a choice function). In *Set Theory: An Introduction to Independence Proofs,* Kunen says,

> If $\mathscr{F} = 0$, then $\bigcup \mathscr{F} = 0$ and $\bigcap \mathscr{F}$ "should be" the set of all sets, which does not exist.

This is at [19, page 13]; Kunen does not define classes until page 23.

I myself am not interested in giving classes the formal existence that they have in so-called von Neumann–Bernays–Gödel set theory, such as is presented by Lemmon [21]. More formalism means more need to check that it agrees with our informal understanding.

For students who just want to collect enough credits to graduate, all of these foundational concerns can be de-emphasized. I used to think it reasonable, on an examination, for me to give a verbal description of a class and to ask the students to give a formula that defines the class. I now think that enough problems can be asked without this. I propose that students should be able to do the following.

1. Add and multiply ordinals in their Cantor normal forms. (Exponentiation is optional.)
2. Recognize equations of ordinals that are identities, and supply inductive proofs.
3. Supply counterexamples to ordinal equations that are not identities, and identify the false steps in proposed proofs that the equations *are* identities. (I recognized late—only in 2015–6—that the students could have difficulty in finding the false steps; but if they cannot do it, they can hardly be said to have learned any mathematics at all.)
4. Perform cardinal computations of Alephs and Beths with addition, multiplication, exponentiation, and suprema.

Questions like, "Prove or disprove: every set is a class" are also standard on my exams.

References

[1] Archimedes. *The Two Books* On the Sphere and the Cylinder, volume I of *The Works of Archimedes.* Cambridge University Press, Cambridge, 2004. Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams, by Reviel Netz.

[2] V. I. Arnol′d. On the teaching of mathematics. *Russian Mathematical Surveys*, 53(1):229–234, 1998.

[3] Alexandre V. Borovik. *Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice.* American Mathematical Society, 2008. Abridged draft 0.99, 4 January 2009.

[4] Cesare Burali-Forti. A question on transfinite numbers. In van Heijenoort [45], pages 104–12. First published 1897.

[5] Georg Cantor. *Contributions to the Founding of the Theory of Transfinite numbers.* Cosimo Classics, New York, 2007. Translation with introduction and notes by Philip E. B. Jourdain of Cantor's 1895 and 1897 papers "Beiträge zur Begründung der transfiniten Mengenlehre." Originally published 1915.

[6] R. G. Collingwood. *The Principles of Art.* Oxford University Press, London, Oxford, and New York, paperback edition, 1958. First published 1938.

[7] R. G. Collingwood. *An Essay on Metaphysics.* Clarendon Press, Oxford, revised edition, 1998. With an Introduction and additional material edited by Rex Martin. Published in paperback 2002. First edition 1940.

[8] R. G. Collingwood. *An Essay on Philosophical Method.* Clarendon Press, Oxford, new edition, 2005. With an Introduction and additional material edited by James Connelly and Giuseppina D'Oro. First edition 1933.

[9] Richard Dedekind. *Essays on the Theory of Numbers. I: Continuity and Irrational Numbers. II: The Nature and Meaning of Numbers.* Dover Publications, New York, 1963. Authorized 1901 translation by Wooster Woodruff Beman of "Stetigkeit und irrationale Zahlen" and "Was sind und was sollen die Zahlen" (second edition, 1893; first, 1887).

[10] Dana Densmore. *Newton's* Principia*: The Central Argument*. Green Lion Press, Santa Fe, New Mexico, third edition, 2003. Translations and Diagrams by William H. Donahue. Completely revised and redesigned. New sections added.

[11] René Descartes. *The Geometry of René Descartes*. Dover Publications, New York, 1954. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

[12] Euclid. *Euclidis Elementa*, volume I of *Euclidis Opera Omnia*. Teubner, Leipzig, 1883. Edited with Latin interpretation by I. L. Heiberg. Books I–IV.

[13] Euclid. *The Thirteen Books of Euclid's* Elements. Dover Publications, New York, 1956. Translated from the text of Heiberg with introduction and commentary by Thomas L. Heath. In three volumes. Republication of the second edition of 1925. First edition 1908.

[14] Abraham A. Fraenkel, Yehoshua Bar-Hillel, and Azriel Levy. *Foundations of set theory*. North-Holland Publishing Co., Amsterdam, revised edition, 1973. With the collaboration of Dirk van Dalen, Studies in Logic and the Foundations of Mathematics, Vol. 67.

[15] Kurt Gödel. On formally undecidable propositions of *principia mathematica* and related systems I. In van Heijenoort [45], pages 596–616. First published 1931.

[16] Timothy Gowers. *Mathematics: A Very Short Introduction*. Very Short Introductions. Oxford University Press, Oxford, 2002.

[17] Herodotus. *The Persian Wars, Books I–II*, volume 117 of *the Loeb Classical Library*. Harvard University Press, Cambridge

MA and London, 2004. Translation by A. D. Godley; first published 1920; revised, 1926.

[18] Immanuel Kant. *Critique of Pure Reason.* The Cambridge Edition of the Works of Kant. Cambridge University Press, Cambridge, paperback edition, 1999. Translated and edited by Paul Guyer and Allen W. Wood. First published 1998.

[19] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1983. An introduction to independence proofs, Reprint of the 1980 original.

[20] Edmund Landau. *Foundations of Analysis. The Arithmetic of Whole, Rational, Irrational and Complex Numbers.* Chelsea Publishing, New York, third edition, 1966. Translated by F. Steinhardt; first edition 1951; first German publication, 1929.

[21] E. J. Lemmon. *Introduction to Axiomatic Set Theory.* Routledge & Kegan Paul / Dover Publications Inc, London / New York, 1969.

[22] Azriel Levy. *Basic set theory.* Dover Publications, Mineola, NY, 2002. Reprint of the 1979 original [Springer, Berlin].

[23] Henry George Liddell and Robert Scott. *A Greek-English Lexicon.* Clarendon Press, Oxford, 1996. "Revised and augmented throughout by Sir Henry Stuart Jones, with the assistance of Roderick McKenzie and with the cooperation of many scholars. With a revised supplement." First edition 1843; ninth edition 1940.

[24] Nicholas Lobachevski. The theory of parallels. In Roberto Bonola, editor, *Non-Euclidean Geometry: A Critical and Historical Study of its Development.* Dover, New York, 1955. Translated by George Bruce Halstead, 1891. Open Court edition, 1914.

[25] Isaac Newton. *The* Principia*: Mathematical Principles of Natural Philosophy.* University of California Press, Berkeley, CA, 1999. A new translation by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz. Preceded by "A guide to Newton's *Principia*" by Cohen.

[26] Pappus of Alexandria. *Pappus Alexandrini Collectionis Quae Supersunt*, volume II. Weidmann, Berlin, 1877. E libris manu scriptis edidit, Latina interpretatione et commentariis instruxit Fridericus Hultsch.

[27] Pappus of Alexandria. *Book 7 of the Collection. Part 1. Introduction, Text, and Translation.* Springer Science+Business Media, New York, 1986. Edited With Translation and Commentary by Alexander Jones.

[28] Pappus of Alexandria. Toplama'nın yedinci kitabı'nda Öklid'in Porizmalar'ı için 38 lemmadan ilk 19 lemma. `mat.msgsu.edu.tr/~dpierce/Courses/Sirince/2016/pappusun-teoremi-2016-09-26.pdf`, September 2016. The First 19 of the 38 Lemmas for Euclid's Porisms in the Seventh Book of the Collection. Turkish translation by David Pierce.

[29] David Pierce. Induction and recursion. *The De Morgan Journal*, 2(1):99–125, 2012. `http://education.lms.ac.uk/2012/04/david-pierce-induction-and-recursion/`.

[30] David Pierce. St John's College. *The De Morgan Journal*, 2(2):62–72, 2012. `education.lms.ac.uk/wp-content/uploads/2012/02/st-johns-college.pdf`, accessed October 1, 2014.

[31] David Pierce. Thales and the nine-point conic. *The De Morgan Gazette*, 8(4):27–78, 2016. `http://education.lms.ac.uk/2016/12/thales-and-the-nine-point-conic/`, accessed June 1, 2017.

[32] David Pierce. On commensurability and symmetry. *J. Humanist. Math.*, 7(2):90–148, 2017. `scholarship.claremont.edu/jhm/vol7/iss2/6`.

[33] Plato. *Republic*. Number 237, 276 in the Loeb Classical Library. Harvard University Press and William Heinemann, Cambridge MA and London, 1982, 1980. with an English Translation by Paul Shorey, in two volumes, first printed 1930 and 1935.

[34] Plato. *Socrates' Defence*. Penguin, UK, 2015. Translated by Christopher Rowe.

[35] Bruno Poizat. *Cours de Théorie des Modèles*. Bruno Poizat, Lyon, 1985. Une introduction à la logique mathématique contemporaine. [An introduction to contemporary mathematical logic].

[36] Bruno Poizat. *A Course in Model Theory*. Universitext. Springer-Verlag, New York, 2000. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author.

[37] Abraham Robinson. *Non-standard analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1996. Reprint of the second (1974) edition. With a foreword by Wilhelmus A. J. Luxemburg. First edition 1965.

[38] Bertrand Russell. Letter to Frege. In van Heijenoort [45], pages 124–5. First published 1902.

[39] S. L. Salas and Einar Hille. *Calculus: One and Several Variables. With Analytic Geometry. Part I.* John Wiley, New York, third edition, 1978.

[40] A. Seidenberg. Did Euclid's Elements, Book I, develop geometry axiomatically? *Arch. History Exact Sci.*, 14(4):263–95, 1975.

[41] Michael Spivak. *Calculus*. Publish or Perish, Berkeley, California, 2nd edition, 1980.

[42] Patrick Suppes. *Axiomatic Set Theory*. Dover Publications, New York, 1972. Unabridged and corrected republication of the 1960 original with a new preface and a new section (8.4).

[43] Alfred Tarski. The concept of truth in formalized languages (1933). In *Logic, semantics, metamathematics*, pages 152–278. Hackett Publishing Co., Indianapolis, IN, second edition, 1983. Papers from 1923 to 1938, Translated by J. H. Woodger, Edited and with an introduction by John Corcoran.

[44] Henry David Thoreau. *Walden*. Princeton University Press, Princeton and Oxford, 2004. 150th anniversary edition. Edited by J. Lyndon Shanley. With an introduction by John Updike. First edition 1971.

[45] Jean van Heijenoort, editor. *From Frege to Gödel: A source book in mathematical logic, 1879–1931*. Harvard University Press, Cambridge, MA, 2002.

[46] John von Neumann. On the introduction of transfinite numbers. In van Heijenoort [45], pages 346–354. First published 1923.

[47] Edward O. Wilson. On free will: And how the brain is like a colony of ants. *Harper's*, pages 49–52, September 2014. Essay.

[48] Ernst Zermelo. Investigations in the foundations of set theory I. In van Heijenoort [45], pages 199–215. First published 1908.