# Linear Algebra Problems

David Pierce

April 2, 2019
Matematik Bölümü, MSGSÜ
mat.msgsu.edu.tr/~dpierce/

Sources include
- Cemal Koç, *Topics in Linear Algebra*
- Steven Roman, *Advanced Linear Algebra* (2d ed.)

**Problem 1.** $X \mapsto \tau_X$, where $\tau_A(Y) = AY$, is an isomorphism from $M$ to $\mathcal{L}(K^n, K^n)$.

**Problem 2.** If $V$ is a finite-dimensional vector space, $\tau$ is an endomorphism of $V$ (in other words, a linear operator on $V$), and $\tau$ and $\tau^2$ have the same rank, show that the kernel and image of $\tau$ have trivial intersection.

**Problem 3.** Supposing $\{\boldsymbol{a}_k : k \in n\}$ is a basis of $V$, define $\tau_{ij}$ in $\mathcal{L}(V)$ by

$$\tau_{ij}(\boldsymbol{a}_k) = \begin{cases} \boldsymbol{a}_k & \text{if } k \neq i, \\ \boldsymbol{a}_i + \boldsymbol{a}_j & \text{if } k = i. \end{cases}$$

(a) Under what conditions on the scalar field is each $\tau_{ij}$ invertible?
(b) Under what conditions on the scalar field do the $\tau_{ij}$ compose a basis of $\mathcal{L}(V)$?

**Problem 4.** Prove concerning determinants:
(a) $\det(\dots, \boldsymbol{v}, \dots, \boldsymbol{v}, \dots) = 0$,
(b) $\det A \cdot \mathrm{I} = A \operatorname{adj}(A)$,
(c) $\det A^{\mathrm{t}} = \det A$.

**Problem 5.** If an abelian group has a multiplication (meaning a binary operation distributing in both senses over addition), and this has both a left and a right identity, show that they are equal.

**Problem 6.** For which $2 \times 2$ matrices is the minimal polynomial the same as the characteristic polynomial?

**Problem 7.** If $A$ is a real matrix and $A^5 + A^3 + A = 0 = A^5 - A^3 - A$, what is the minimal polynomial of $A$?

**Problem 8.** Prove that the characteristic polynomial of every real $2 \times 2$ symmetric matrix is the product of linear factors.

**Problem 9.** Show that $AB$ and $BA$ have the same eigenvalues.

**Problem 10.** If $A$ is diagonalizable, show the same for every polynomial function $f(A)$ of $A$.

**Problem 11.** If $A$ is **nilpotent,** that is, some power of $A$ is the zero matrix, show $A$ is not diagonalizable.

**Problem 12.** Find the Jordan normal form of every $5 \times 5$ nilpotent matrix.

**Problem 13.** Find the Jordan normal form of every $n \times n$ complex matrix of which some positive power is I.