Ordinal Analysis II

A course at the Nesin Matematics Village

David Pierce

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Matematik Bölümü
Mimar Sinan Güzel Sanatlar Üniversitesi
İstanbul
dpierce@msgsu.edu.tr
mat.msgsu.edu.tr/~dpierce/
polytropy.com

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Preface

Here are notes in English from Ordinal Analysis II, given at the Nesin Matematik Köyü in Şirince, February 3–9, 2020. English is the language I wrote on the blackboards of the Nişanyan Kütüphanesi, while speaking mostly in Turkish. One student proposed that I speak in English, but another student objected.

At the beginning of the first day, when there were around fifty students, I wrote the sequence of ordinals—(1) on page 9—across all four boards in the library. On the last day, there were seven students.

I had given Ordinal Analysis I in the previous week, in the Arf Dersliği. The two courses together together covered much of what I had done in Aksiyomatik Kümeler Kuramı in the previous semester at Mimar Sinan.

Ordinal Analysis I introduced the Zermelo–Fraenkel axioms as they were needed to develop and prove the desired properties of the class **ON** of ordinals.

Ordinal Analysis II assumed those properties (listed on page 23) and developed the arithmetic of the ordinals.

The latter course is summarized on page 8. I did not typeset new notes for the first course. I did not assume that students in the second course had taken the first. In all but one or two cases they had not. One student from the Mimar-Sinan course did attend the entirety of Ordinal Analysis II.

I started that course, thinking that I would progressively introduce concrete examples of well-ordered sets:

1) ω, closed under addition, multiplication, and exponentiation, but containing no limits;
2) $\omega \sqcup \omega$, isomorphic to $\omega + \omega$ and containing a single limit;
3) $\omega \times \omega$, isomorphic to $\omega \cdot \omega$, closed under addition, and containing infinitely many limits, which themselves have no limit;
4) the set of operations on ω that have finite support: this set is isomorphic to ω^ω, is closed under addition and multiplication, and has limits of limits;
5) the set of finite rooted trees: this set is isomorphic to ε_0 and closed under all of the operations.

In the event, I did not isolate the last two examples, but I interpreted arbitrary powers α^β as sets of functions, and after ω I considered only **ON** itself as being closed under all of the operations.

I often left proofs as exercises. Had there been time, I would have had students give their proofs at the board: I do this at Mimar Sinan, and I have done it in other courses at the Math Village. This time I wanted to cover all of my Mimar-Sinan course in two-thirds of the time.

Some students asked for references. These would include:

- the notes (in Turkish) from my Mimar-Sinan course;
- any set-theory textbook, particularly Levy, *Basic Set Theory* (Dover, 2002), which has perhaps been my own main reference;
- Ali Nesin's Açık Ders text, *Aksiyomatik Kümeler Kuramı (Dönem 1),* on the website of the Türkiye Bilimler Akademisi. I have recommended this text to students at Mimar Sinan and used as a source for Turkish terminology.

My Personal Journey

I was not prepared for the question of references. Since I started learning real mathematics from one Donald J. Brown in my last two years (1981–3) at St. Albans School in Washington, D.C., I have understood the main text of a mathematics course to be the teacher's own lectures. Mr Brown had us copy down exactly what he wrote on the blackboards of his classroom, and he inspected our notebooks (though perhaps mainly to see that we had reinforced the holes in our sheets of paper so that they would not easily tear from our three-ring binders).

Mr Brown did have us buy supplementary books, to be used particularly as sources of exercises.* One of the books was Spivak's *Calculus.* I read this, while understanding that Spivak's

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*For the record, the books were the following, roughly in order of use.

- Kutepov and Rubanov, *Problem Book: Algebra and Elementary Functions* (Moscow: Mir, 1978);
- Dorofeev, Potapov, and Rozov, *Elementary Mathematics: Selected Topics and Problem Solving* (Moscow: Mir, 1973);
- Spivak, *Calculus,* Second Edition (Berkeley: Publish or Perish, 1980);
- Salas and Hille, *Calculus: One and Several Variables, Part I,* Third Edition (New York: Wiley, 1978);
- Petit Bois, *Tables of Indefinite Integrals* (New York: Dover, 1961);
- Apostol, *Mathematical Analysis,* Second Edition (Reading, Massachusetts: Addison-Wesley, 1974).

I am able to list these books, because I have kept them.

approach to the subject was not "official" when it differed from our own (as it did for example in the development of the real numbers as a complete ordered field, and later in the definitions of the trigonometric functions). I might have preferred Spivak's or another approach; If I ever taught my own course. I could do things my way.

This is what I have often done, starting when I was assigned to join two colleagues in teaching Fundamentals of Mathematics at METU in Ankara in the fall of 2001-2. One of those colleagues (who was from Germany) proposed that he and I write the text of the course. Part of my own share of the writing became the text that Ayşe Berkman used when teaching Set Theory at METU in the fall of 2003-4. I revised and rewrote that text when teaching the set-theory course for myself in 2006-7, 2008-9, and 2010-1. When I moved to Istanbul, I prepared texts in Turkish for the course Aksiyomatik Kümeler Kuramı, which I have given so far in 2012-3, 2013-4, 2015-6, 2017-8, and 2019-20.

Without going back and reviewing all of those texts (though they are all on my webpage at Mimar Sinan), I think the progression has been as follows. Initially I tried to write down what a *teacher* should know, or at least what *I* wanted to know. I pared down down what I wrote as I learned the material and the students. I have come to emphasize ordinal arithmetic, through computation with Cantor normal forms, because, as I understand it, the Turkish national university entrance examinations are largely computational, at least as far as mathematics is concerned.

It has also come to seem worthwhile to me to develop an analogy between the "well-order" of the ordinal numbers and the complete dense ordering of the real numbers. I am not a set theorist in the sense of publishing new research in the subject, and I do not expect my students to become set theorists (though one of them has). Students are more likely to work with analysis than set theory. The former had been my own introduction to modern mathematical rigor, through Mr Brown's two-year course of honors precalculus and calculus. All students of mathematics learn such techniques as differentiation and integration, which are justified by the aforementioned rigor. Students may forget the techniques, but they ought to retain an awareness of the possibility of justifying them. The standards of justification are universal, precisely because they are the individual property of each of us.

Summary

Monday Paradoxes of Russell and Burali-Forti. ω and $\omega \sqcup \omega$ as well-ordered sets. Addition on ω is defined by recursion; its properties are proved by induction. Addition of a natural number to an element of $\omega \sqcup \omega$ in two ways, only one being continuous.

Tuesday Continuity of functions, especially increasing functions, from ordered sets to ordered sets.

Wednesday The desired properties of $\mathbb{R}$ lead to Dedekind's construction; of **ON**, to von Neumann's. Transfinite recursion. $\alpha + \beta \cong \alpha \sqcup \beta$.

Thursday History of our subject. Properties of addition and multiplication of ordinals proved by transfinite induction. An increasing operation on **ON** is continuous if and only if its value at every limit is the supremum of its values at the preceding ordinals. $\alpha \cdot \beta \cong \alpha \times \beta$. Cantor normal forms of elements of $\omega \times \omega$ and ω^{ω}.

Friday Ordinal exponentiation. Cantor normal forms of arbitrary ordinals and the generalization to arbitrary bases. α^{β} is isomorphic to the set of functions from β to α with finite support.

Saturday Computation with Cantor normal forms. Each of $\alpha + \beta$, $\alpha \cdot \beta$, and α^{β} is equipollent with the greater of α and β, if these are infinite.

1 Monday

The *ordinal numbers* or **ordinals** compose this *linear order*:*

$$\left.\begin{array}{c} 0, 1, 2, \dots; \omega, \omega+1, \omega+2, \dots; \omega+\omega = \omega\cdot 2, \\ \omega\cdot 2+1, \dots; \omega\cdot 3, \omega\cdot 3+1, \dots; \omega\cdot\omega = \omega^2, \\ \omega^2+1, \dots; \omega^2+\omega, \omega^2+\omega+1, \dots; \omega^3, \dots; \omega^\omega, \\ \omega^\omega+1, \dots, \omega^{\omega^\omega}, \dots; \varepsilon_0, \varepsilon_0+1, \dots; \varepsilon_1, \dots; \omega_1, \dots \end{array}\right\} \tag{1}$$

- ω is *omega*, "large o," the minuscule case of the last letter of the Greek alphabet; we shall understand

  $$\omega = \{0, 1, 2, \dots\},$$

  the set of **natural numbers.** (The set $\{1, 2, 3, \dots\}$ of *counting numbers* can be denoted by $\mathbb{N}$.)
- ε_0 will be the first solution; ε_1, the next; to the equation

  $$\omega^x = x.$$

- ω_1 will be the least ordinal α such that the set

  $$\{x\colon x < \alpha\}$$

  is **uncountable,** meaning there is no bijection with ω or a subset of it.†

*The definition of linear ordering is spelled out on page 23.
†People asked about the ε_x and ω_1, so I talked about them.

Moreover, the ordinals have the following properties:

1. Every α of them has a **successor,** α' or $\alpha + 1$, which is the first of those that are greater than it.
2. Every set A of them that has no greatest element still has a **supremum** or least upper bound, $\sup(A)$.

Theorem 1 (Burali-Forti Paradox)**.** *The class of ordinals is not a set.*

Proof. Since every ordinal has a successor, the class of ordinals has no supremum, so it cannot be a set. □

The class of ordinals is "too big" to be a set. Another example is given by the following.

Theorem 2 (Russell Paradox)**.** *The class* $\{x\colon x \notin x\}$ *is not a set.*

A technical property of the ordinals is the following.

3. They are **left-narrow:**[*] for each α of them, the class $\{x\colon x < \alpha\}$ or $\mathrm{pred}(\alpha)$ is a set.

Theorem 3. *The class of ordinals is* ***well-ordered:*** *it is left-narrow, and every nonempty set* A *of them has a least element,* $\min(A)$.

Proof. Let B be a nonempty set of ordinals, and let

$$A = \bigcap_{x \in B} \mathrm{pred}(x) = \{x\colon \forall y\ (y \in B \Rightarrow y < x)\}.$$

This is a set of ordinals, so it has a supremum, α.

- If $\alpha \notin A$, then it is the least element of B.

[*]In class, I did not use this term, which is from Levy, I.1(vii), p. 33.

- If $\alpha \in A$, then its successor is the least element of B. □

Conversely, the following are true in every well-ordered class.

1. If α is not the greatest element, then
$$\alpha' = \min\{x\colon \alpha < x\}.$$
2. If A has an upper bound, then
$$\sup(A) = \min\{x\colon \forall y\ (y \in A \Rightarrow y \leqslant x)\}.$$

In ω, every element is either the least element, namely 0, or the successor of an element.

Theorem 4. ω *admits* ***(finite) induction:*** *If* $A \subseteq \omega$ *and*

1) $0 \in A$,

2) $\forall x\ (x \in A \Rightarrow x' \in A)$,

then $A = \omega$.

Proof. The least element $\omega \smallsetminus A$ can be neither 0 nor a successor, so there isn't one. Therefore $\omega \smallsetminus A = \varnothing$. □

For any sets A and B, we define
$$A \sqcup B = (A \times \{0\}) \cup (B \times \{1\}),$$
the **disjoint union** of A and B.

Theorem 5. $\omega \sqcup \omega$ *is well-ordered by the rule*
$$(a, e) < (b, f) \Leftrightarrow e < f \vee (e = f \wedge a < b).$$

Proof. In order,
$$\omega \sqcup \omega = \{(0,0), (1,0), (2,0), \dots; (0,1), (1,1), (2,1), \dots\}.$$
Thus $\omega \sqcup \omega$ is well-ordered. Indeed, for nonempty subsets A,
$$\min(A) = \begin{cases} (\min\{x\colon (x,0) \in A\}, 0), & \text{if } \exists x\ (x,0) \in A; \\ (\min\{x\colon (x,1) \in A\}, 1), & \text{if } \forall x\ (x,0) \notin A. \end{cases}$$
□

In $\omega \sqcup \omega$, $(0, 1)$ is neither the least element nor the successor of an element. We call such an element of a well-ordered set a **limit.** Thus a limit is an element a such that

$$\exists x\ x < a \wedge \forall x\ (x < a \Rightarrow x' < a).$$

Let us denote this condition by

$$\lim(a).$$

Theorem 6. *$\omega \sqcup \omega$ admits* ***transfinite induction:*** *If $A \subseteq \omega \sqcup \omega$ and*

1) $0 \in A$,
2) $\forall x\ (x \in A \Rightarrow x' \in A)$,
3) $\forall x\ (\lim(x) \wedge \mathrm{pred}(x) \subseteq A \Rightarrow x \in A)$,

then $A = \omega$.

Proof. The least element $(\omega \sqcup \omega) \smallsetminus A$ can be neither 0, nor a successor, nor a limit. □

Theorem 7. *On ω, functions can be defined by* ***(finite) recursion:*** *If A is any set, and $b \in A$, and $f : A \to A$, then a unique function g from ω to A exists such that*

1) $g(0) = b$,
2) $\forall x\ g(x') = f(g(x))$.

Proof. If it exists, such a function is unique, by induction. Likewise, for each n in ω, if it exists, a function g_n from $\mathrm{pred}(n')$ to A such that

1) $g_n(0) = b$;
2) $\forall x\ (x < n \Rightarrow g(x') = f(g(x))$,

is unique, by induction. By induction, g_n does exist for all n in ω, since

1) g_0 can be $\{(0, b)\}$,

2) $g_{n'}$ can be $g_n \cup \{(n', f(g_n(n)))\}$.
Now we can define
$$g(n) = g_n(n). \qquad \square$$

Now for each n in ω we define the operation
$$x \mapsto n + x$$
on ω by the rules
1) $n + 0 = n$,
2) $n + k' = (n + k)'$.

Now we have **addition** as a binary operation on ω.

Theorem 8. *For each n in ω, the operation $x \mapsto n + x$ is strictly increasing.*

Proof. We prove
$$k < m \Rightarrow n + k < n + m$$
by induction on the rightmost letter, m.

1. Since always $k \not< 0$, we conclude
$$k < 0 \Rightarrow n + k < n + 0.$$
2. Suppose $k < \ell \Rightarrow n + k < n + \ell$. Say now $k < \ell'$. Then $k \leqslant \ell$. There are two cases.
   a) If $k = \ell$, then $n + k = n + \ell$.
   b) If $k < \ell$, then $n + k < n + \ell$ by hypothesis.

   In either case,
$$\begin{aligned} n + k \leqslant n + \ell &< (n + \ell)' && \text{[definition]} \\ &= n + \ell'. && \text{[definition]} \end{aligned} \qquad \square$$

Theorem 9. *Addition on ω is associative.*

Proof. We prove

$$(n+k)+m=n+(k+m)$$

by induction on m.*

1. By definition,

$$(n+k)+0=n+k=n+(k+0).$$

2. Suppose $(n+k)+\ell=n+(k+\ell)$. Then

$$\begin{aligned}(n+k)+\ell' &= ((n+k)+\ell)' && \text{[definition]}\\ &= (n+(k+\ell))' && \text{[hypothesis]}\\ &= n+(k+\ell)' && \text{[definition]}\\ &= n+(k+\ell'). && \text{[definition]}\end{aligned}$$ □

Lemma 1. *On ω,*

$$0+n=n.$$

Proof. We use induction.†

1. By definition,

$$0+0=0.$$

2. Suppose $0+m=m$. Then

$$\begin{aligned}0+m' &= (0+m)' && \text{[definition]}\\ &= m'. && \text{[hypothesis]}\end{aligned}$$ □

Lemma 2. *On ω,*

$$(n+k)'=n'+k.$$

*I did not give the remainder of the proof.

†In class I combined this and the next lemma as one, leaving the proof of the first part as an exercise.

Proof. We use induction on k.

1. By definition,

$$(n+0)' = n' = n' + 0.$$

2. Suppose $(n+m)' = n' + m$. Then

$$\begin{aligned}(n+m')' &= (n+m)'' && \text{[definition]}\\ &= (n'+m)' && \text{[hypothesis]}\\ &= n' + m'. && \text{[definition]}\end{aligned}$$

□

Theorem 10. *Addition is commutative on* ω*.*

Proof. We prove $n + k = k + n$ by induction on n. First,

$$\begin{aligned}0 + k &= k && \text{[Lemma 1]}\\ &= k + 0. && \text{[definition]}\end{aligned}$$

Next, if $m + k = k + m$, then

$$\begin{aligned}m' + k &= (m+k)' && \text{[Lemma 2]}\\ &= (k+m)' && \text{[hypothesis]}\\ &= k + m'. && \text{[definition]}\end{aligned}$$

□

Theorem 11. *For each* n *in* ω*, the function* $x \mapsto x + n$ *is strictly increasing.*

Proof. Theorems 8 and 10. □

We can embed ω in $\omega \sqcup \omega$ by the rule

$$x \mapsto (x, 0).$$

Then we can identify each element of ω with its image, but define

$$\omega + n = (n, 1).$$

In order then,

$$\omega \sqcup \omega = \{0, 1, 2, \dots; \omega, \omega + 1, \omega + 2, \dots\}.$$

For each k in ω, we extend the operation

$$x \mapsto x + k$$

from the domain ω to the domain $\omega \sqcup \omega$ by defining

$$(\omega + n) + k = \omega + (n + k).$$

The operation is still strictly increasing. However,

$$\sup\{x + k \colon x < \omega\} = \omega,$$

which is strictly less than $\omega + k$ if $k > 0$. Therefore, as we shall see, the operation then is not *continuous* at ω. If we define

$$x \mapsto k + x$$

on $\omega \sqcup \omega$ by

$$k + (\omega + n) = \omega + n,$$

then the operation is still strictly increasing. Also,

$$\sup\{k + x \colon x < \omega\} = \omega = k + \omega,$$

so the operation *is* continuous at ω.

2 Tuesday

To define continuity, suppose f is a function from a linear order A to a linear order B, and $c \in A$. For example, A and B could be the set of real numbers. Then f is **continuous** at c, provided that for all ε_1 and ε_2 in B such that

$$\varepsilon_1 < f(c) < \varepsilon_2,$$

for some δ_1 and δ_2 in A such that

$$\delta_1 < c < \delta_2,$$

for all x in A,

$$\delta_1 < x < \delta_2 \Rightarrow \varepsilon_1 < f(x) < \varepsilon_2.$$

In case B or A has a minimum or maximum, and $f(c)$ or c is it, we must allow

- ε_1 or δ_1 to be $-\infty$,
- ε_2 or δ_2 to be ∞.

In case B is well-ordered, we shall understand

$$(-\infty)' = \min(B), \qquad \max(B)' = \infty,$$

and likewise for A.

Theorem 12. *Suppose A and B are well-ordered, $f\colon A \to B$, and $c \in A$.*

1. *If c is the least element or a successor in A, then f is continuous at c.*
2. *Suppose c is a limit. The following are equivalent.*
   a) *f is continuous at c.*
   b) *For all ε such that $\varepsilon < f(c)$, for some δ such that $\delta < c$, for all x in A,*

$$\delta < x < c \Rightarrow \varepsilon < f(x) \leqslant f(c).$$

   c) *For all ε such that $\varepsilon < f(c)$, for some δ such that $\delta < c$,*

$$\varepsilon < \min_{\delta<x<c} f(x), \qquad \sup_{\delta<x<c} f(x) = f(c).$$

Proof. 1. If c is not a limit in A, we may suppose $c = d'$ for some d in $\{-\infty\} \cup A$. Also $c' \in A \cup \{\infty\}$. Thus

$$d < c < c'.$$

Moreover,

$$\forall x\ (d < x < c' \Rightarrow x = c).$$

Thus for all ε_1 and ε_2 in B such that

$$\varepsilon_1 < f(c) < \varepsilon_2,$$

for all x in A,

$$d < x < c' \Rightarrow \varepsilon_1 < f(x) < \varepsilon_2.$$

Therefore f is continuous at c.

2. Now suppose c is a limit.

a) Suppose f is continuous at c, and $\varepsilon < f(c)$. For some δ such that $\delta < c$, for all x in A,

$$\delta < x < c' \Rightarrow \varepsilon < f(x) < f(c)'$$

and therefore

$$\delta < x \leqslant c \Rightarrow \varepsilon < f(x) \leqslant f(c).$$

b) Suppose that, for all ε such that $\varepsilon < f(c)$, for some δ such that $\delta < c$,

$$\forall x\ (\delta < x < c \Rightarrow \varepsilon < f(x) \leqslant f(c)).$$

This means precisely

$$\varepsilon < \min_{\delta<x<c} f(x), \qquad \sup_{\delta<x<c} f(x) \leqslant f(c).$$

Suppose if possible that, for all δ such that $\delta < c$,

$$\sup_{\delta<x<c} f(x) < f(c).$$

Now let $\varepsilon = \sup_{\delta<x<c} f(x)$. By our hypothesis, for some d such that $d < c$,

$$\forall x\ (d < x < c \Rightarrow \varepsilon < f(x) \leqslant f(c)).$$

Since c is a limit, there is a in A such that $\max(d, \delta) < a < c$. Then

$$f(a) \leqslant \sup_{\delta<x<c} f(x) = \varepsilon < f(a),$$

which is absurd.

c) Suppose for all ε such that $\varepsilon < f(c)$, for some δ such that $\delta < c$,

$$\varepsilon < \min_{\delta<x<c} f(x), \qquad \sup_{\delta<x<c} f(x) = f(c).$$

Then

$$\forall x\ (\delta < x < c' \Rightarrow \varepsilon < f(x) \leqslant f(c) < f(c)'.$$

Thus f is continuous at c. □

Corollary. *Under the conditions of the theorem, if f is also increasing, then it is continuous at a limit c if and only if*

$$\sup_{x<c} f(x) = f(c).$$

The set $\omega \sqcup \omega$ is $\omega \times \{0, 1\}$, a subset of $\omega \times \omega$. This is well-ordered by the rule of Theorem 5. Let us use the notation

$$\omega \cdot n + k = (k, n).$$

The elements $\omega \cdot n$ are limits when $n > 0$. Now we can define

$$(\omega \cdot n + k) + (\omega \cdot m + \ell) = \begin{cases} \omega \cdot n + (k + \ell), & \text{if } m = 0, \\ \omega \cdot (n + m) + \ell, & \text{if } m > 0. \end{cases} \tag{2}$$

Theorem 13. *For each α in $\omega \times \omega$, the operation*

$$x \mapsto \alpha + x$$

is strictly increasing and continuous; the operation

$$x \mapsto x + \alpha$$

is increasing, though not strictly, and is not continuous unless $\alpha = 0$.

We could also define

$$(\omega \cdot n + k) \oplus (\omega \cdot m + \ell) = \omega \cdot (n + m) + (k + \ell).$$

Then $\oplus$ is commutative and associative, but continuous in neither argument.

3 Wednesday

What *is* an ordinal number? Well, what is an real number? To do real analysis, all one need know is that the real numbers together compose a *complete ordered field,* $\mathbb{R}$. That this is **complete** means every nonempty subset with an upper bound has a least upper bound, or supremum.

Every ordered field includes the ordered field $\mathbb{Q}$ of rational numbers. In an ordered field, if $\mathbb{Q}$ has an upper bound a, then $a - 1$ is also an upper bound; therefore there is no least upper bound.

Ordered fields in which $\mathbb{Q}$ is bounded do exist. For example, if

$$\mathbb{Q}[X] = \{\text{polynomials in } X \text{ over } \mathbb{Q}\},$$
$$\mathbb{Q}(X) = \left\{\frac{f}{g} \colon f \in \mathbb{Q}[X] \land g \in \mathbb{Q}[X] \smallsetminus \{0\}\right\},$$

this field can be ordered by the rule

$$\frac{a_0 + a_1 X + \cdots + a_m X^m}{b_0 + b_1 X + \cdots + b_n X^n} > 0 \Leftrightarrow \frac{a_m}{b_n} > 0.$$

Hence for all a in $\mathbb{Q}$, $X - a > 0$, and so $X > a$. Thus X is an upper bound of $\mathbb{Q}$.

An ordered field in which $\mathbb{Q}$ has no upper bound is called **Archimedean.** Being complete, $\mathbb{R}$ must be Archimedean.

Because of this, $\mathbb{Q}$ is **dense** in $\mathbb{R}$: between any two real numbers lies a rational number. Therefore the function f on $\mathbb{R}$ given by

$$f(\alpha) = \{x \in \mathbb{Q} \colon x < \alpha\}$$

is injective. Moreover, its range is the set of subsets A of $\mathbb{Q}$ such that

$$\varnothing \subset A \subset \mathbb{Q},$$
$$\forall x \, \forall y \, (x < y \land y \in A \Rightarrow x \in A).$$

Thus we can define the range of f without having f or $\mathbb{R}$. This means we can *define* $\mathbb{R}$ as the range of f.

One then must extend the definitions of addition and multiplication on $\mathbb{Q}$ to $\mathbb{R}$. For every a in $\mathbb{Q}$, the functions $x \mapsto a+x$ and $x \mapsto a \cdot x$ are continuous on $\mathbb{Q}$; they will remain so on $\mathbb{R}$.

To do ordinal analysis, all we need know is that the ordinal numbers together compose a class **ON** with certain properties:

1. **ON** is **well-ordered.** This means:
   a) **ON** is **linearly ordered,** so that for any α, β, and γ in **ON**,

   $$\alpha \not< \alpha,$$
   $$\alpha < \beta \land \beta < \gamma \Rightarrow \alpha < \gamma,$$
   $$\alpha \leqslant \beta \lor \alpha > \beta.$$

   b) every nonempty subset A of **ON** has a least element, $\min(A)$.
   c) the class $\operatorname{pred}(\alpha)$ or $\{x \colon x < \alpha\}$ of predecessors of a given ordinal α is a set.*

*It follows now that every nonempty *class* of ordinals has a least element, but I did not spell this out.

2. **ON** is non-empty, so it has a least element, 0.
3. **ON** has no greatest element, so every element α of **ON** has a successor, α', namely $\min\{x \colon x > \alpha\}$.
4. Every subset A of **ON** has an upper bound and therefore a supremum, $\sup(A)$, namely

$$\min\{x \colon \forall y\ (y \in A \Rightarrow y \leqslant x)\}.$$

5. **ON** contains limits, namely elements that are neither 0 nor successors; the least limit is ω.

Can we prove that there are more limits than ω? As we did from ω to $\omega \sqcup \omega$, we recursively define $x \mapsto \omega + x$ from the subset $\mathrm{pred}(\omega)$ of **ON** to **ON** itself by

$$\omega + 0 = \omega, \qquad\qquad \omega + x' = (\omega + x)'.$$

Then $\sup\{\omega + x \colon x < \omega\}$ is the next limit after ω. It exists by the **Replacement Axiom:** the image of a set under a function is still a set.

For any α in **ON**, we define $x \mapsto \alpha + x$ by *transfinite recursion*:

1. $\alpha + 0 = \alpha$.
2. $\alpha + \beta' = (\alpha + \beta)'$.
3. $\lim(\gamma) \Rightarrow \alpha + \gamma = \sup\{\alpha + x \colon x < \gamma\}$.

We can do this by the following.

Theorem 14 (Transfinite Recursion). *If $\alpha \in$ **ON** and $\boldsymbol{F}$ is an operation on **ON**, then a unique operation $\boldsymbol{G}$ on **ON** exists such that*

1) $\boldsymbol{G}(0) = \alpha$,
2) $\forall x\ (\boldsymbol{G}(x') = \boldsymbol{F}(\boldsymbol{G}(x)))$,
3) $\forall x\ (\lim(x) \Rightarrow \boldsymbol{G}(x) = \sup(\boldsymbol{G}[\mathrm{pred}(x)]))$.

Proof. By transfinite induction, there is at most one such $\boldsymbol{G}$. For the same reason, for each β in **ON**, there is at most one function g_β from $\operatorname{pred}(\beta')$ to **ON** with the properties of $\boldsymbol{G}$, and also

$$\delta \leqslant \gamma \leqslant \beta \Rightarrow g_\beta(\delta) = g_\gamma(\delta).$$

Again by transfinite induction, there is at least one such g_β for each β; for we can let

$$g_0 = \{(0, \alpha)\},$$
$$g_{\beta'} = g_\beta \cup \{(\beta', \boldsymbol{F}(g_\beta(\beta)))\},$$
$$\lim(\gamma) \Rightarrow g_\gamma = \bigcup\{g_x \colon x < \gamma\} \cup \{(\gamma, \sup\{g_x(x) \colon x < \gamma\})\}.$$

Then $\boldsymbol{G}$ can be $x \mapsto g_x(x)$. □

By a modification of the proof, there is $\boldsymbol{G}$ such that
1) $\boldsymbol{G}(0) = \varnothing$,
2) $\forall x\ \boldsymbol{G}(x') = \boldsymbol{G}(x) \cup \{\boldsymbol{G}(x)\}$,
3) $\forall x\ (\lim(x) \Rightarrow \boldsymbol{G}(x) = \boldsymbol{G}[\operatorname{pred}(x)])$.

In this case,

$$\forall x\ (\boldsymbol{G}(x) = \boldsymbol{G}[\operatorname{pred}(x)]).$$

Then $\boldsymbol{G}[\mathbf{ON}]$ consists of sets that are
- well-ordered by $\in$,
- *transitive,*

where a set A is **transitive** if

$$\forall x\ \forall y\ (y \in x \land x \in A \Rightarrow y \in A),$$

that is,

$$\forall x\ (x \in A \Rightarrow x \subseteq A).$$

One can show that *every* transitive set that is well-ordered by $\in$ belongs to the range of $\boldsymbol{G}$. Thus we can define **ON** as this range. Now each α in **ON** is $\operatorname{pred}(\alpha)$; also,

$$\alpha' = \alpha \cup \{\alpha\}.$$

Hence

$$0 = \varnothing, \qquad 1 = 0' = \{0\}, \qquad 2 = 1' = \{0, 1\},$$

and so on. Also, if A is a set of ordinals, then

$$\sup(A) = \bigcup A = \{x \colon \exists y\ (y \in A \wedge x \in y)\}.$$

Theorem 15. *For all α and β in* **ON**,

$$\alpha + \beta \cong \alpha \sqcup \beta,$$

where $\alpha \sqcup \beta$ is well-ordered by the rule,

$$(a, b) < (c, d) \Leftrightarrow b < d \vee (b = d \wedge a < c).$$

Proof. Since $\alpha + \beta = \alpha \cup \{\alpha + x \colon x < \beta\}$, the map

$$x \mapsto \begin{cases} (x, 0), & \text{if } x < \alpha, \\ (y, 1), & \text{if } x = \alpha + y < \alpha + \beta \end{cases}$$

is an order-preserving bijection from $\alpha + \beta$ to $\alpha \sqcup \beta$. □

We know from yesterday that $x \mapsto \alpha + x$ is always continuous. What is the least positive ordinal, after ω,* on which $(x, y) \mapsto x + y$ is closed?

*I forgot this condition in class.

4 Thursday

Some history:
Archimedes (died 212 B.C.E.): examples of integral calculus
Newton, Leibniz (born 1642, 1646): calculus
Dedekind (b. 1831): definition of $\mathbb{R}$ (discovered 1858, published 1872)
Cantor (b. 1845): ordinals beyond ω (1869)
Zermelo (b. 1871): most of our set-theory axioms (1908)
Skolem, Fraenkel (b. 1887, 1891): the Replacement Axiom (1922)
von Neumann (b. 1903): definition of **ON** (1923)

Yesterday we defined addition on **ON** by transfinite recursion. We proved the following on Monday, as Theorem 8, for addition on ω.

Theorem 16. *For each α in* **ON**, *the operation $x \mapsto \alpha + x$ on* **ON** *is strictly increasing.*

Proof. We prove

$$\beta < \gamma \Rightarrow \alpha + \beta < \alpha + \gamma$$

by transfinite induction on γ. The first and limit steps are just as for Theorem 8:

1. The claim is true when $\gamma = 0$.
2. If, for some δ, the claim is true when $\gamma = \delta$, then it is true when $\gamma = \delta'$.

For the third step, the limit step, we suppose δ is a limit and the claim is true whenever $\gamma < \delta$. Suppose also $\beta < \delta$. Then

$$\beta < \gamma < \delta$$

for some γ (for example, $\gamma = \beta'$). Therefore

$$\alpha + \beta < \alpha + \gamma \leqslant \sup_{x<\delta}(\alpha + x) = \alpha + \delta,$$

so our claim is true when $\gamma = \delta$. This completes the proof by transfinite induction. □

Now $x \mapsto \alpha + x$ is also continuous, by Theorem 12 and its corollary from Tuesday: if an operation $\boldsymbol{F}$ on **ON** (or from any well-ordered set or class to a well-ordered set or class) is increasing, then $\boldsymbol{F}$ is continuous if and only if, at every limit γ,

$$\boldsymbol{F}(\gamma) = \sup_{x<\gamma} \boldsymbol{F}(x) = \sup \boldsymbol{F}[\gamma].$$

We proved the following as Theorem 9, for addition on ω.

Theorem 17. *Addition on* **ON** *is associative.*

Proof. We prove

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

by transfinite induction on γ. The first two steps are just as for Theorem 9. Suppose now δ is a limit, and the claim is true whenever $\gamma < \delta$. Then

$$\begin{aligned}
\alpha + (\beta + \delta) &= \alpha + \sup_{x<\delta}(\beta + x) && \text{[definition]}\\
&= \sup_{x<\delta}(\alpha + (\beta + x)) && \text{[Theorem 18]}\\
&= \sup_{x<\delta}((\alpha + \beta) + x) && \text{[hypothesis]}\\
&= (\alpha + \beta) + \delta. && \text{[definition]}
\end{aligned}$$

□

Here we have used the following.

Theorem 18. *Suppose*

- $\boldsymbol{F}$ *is an increasing, continuous operation on* $\mathbf{ON}$,
- A *is a nonempty subset of* $\mathbf{ON}$,
- $\alpha = \sup(A)$.

1. *If* $\alpha \notin A$, *then*

$$\sup \boldsymbol{F}[A] = \sup \boldsymbol{F}[\alpha].$$

2. *In any case,*

$$\boldsymbol{F}(\alpha) = \sup \boldsymbol{F}[A].$$

Proof. The second part is easy when $\alpha \in A$. Suppose $\alpha \notin A$. Then α must be a limit, so, since $\boldsymbol{F}$ is increasing and continuous,

$$\boldsymbol{F}(\alpha) = \sup \boldsymbol{F}[\alpha].$$

The second part now follows from the first. Since $A \subseteq \alpha$ and $\boldsymbol{F}$ is increasing,

$$\sup \boldsymbol{F}[A] \leqslant \sup \boldsymbol{F}[\alpha].$$

When $\beta < \alpha$, then $\beta < \gamma$ for some γ in A, and therefore, again since $\boldsymbol{F}$ is increasing,

$$\sup \boldsymbol{F}[\alpha] \leqslant \sup \boldsymbol{F}[A].$$ □

Note that

$$\alpha + 1 = \alpha + 0' = (\alpha + 0)' = \alpha'.$$

Thus

$$1 + \omega = \sup_{x<\omega}(1 + x) = \omega < \omega' = \omega + 1,$$

so addition is not commutative; likewise, for all n in ω,

$$n + \omega = \omega.$$

The following generalizes Lemma 1 (used to prove Theorem 10, that addition is commutative on ω).

Theorem 19. *For all* α *in* **ON**,

$$0 + \alpha = \alpha.$$

Proof. Transfinite induction. □

The following is the version of Theorem 11 (that $x \mapsto x + n$ is strictly increasing on ω) for **ON**.

Theorem 20. *For all* α *in* **ON**, *the operation* $x \mapsto x + \alpha$ *is increasing, that is,*

$$\beta < \gamma \Rightarrow \beta + \alpha \leqslant \gamma + \alpha.$$

Proof. Transfinite induction on α. □

It follows from Theorems 16 and 20 that α is closed under addition if (and only if) it is closed under $x \mapsto x + x$. This operation is $x \mapsto x \cdot 2$ by the following definition.

1. $\alpha \cdot 0 = 0$.
2. $\alpha \cdot \beta' = \alpha \cdot \beta + \alpha$.
3. $\lim(\gamma) \Rightarrow \alpha \cdot \gamma = \sup\{\alpha \cdot x \colon x < \gamma\}$.

By Theorem 19,

$$\alpha \cdot 1 = \alpha \cdot 0' = \alpha \cdot 0 + \alpha = 0 + \alpha = \alpha,$$
$$\alpha \cdot 2 = \alpha \cdot 1' = \alpha \cdot 1 + \alpha = \alpha + \alpha,$$

and so on.

Theorem 21. *For all* α *in* **ON**,

1) $0 \cdot \alpha = 0$,
2) $1 \cdot \alpha = \alpha$.

Proof. Transfinite induction. □

The operation $x \mapsto \alpha \cdot x$ is now

- increasing and continuous when $\alpha = 0$;
- strictly increasing and continuous when $\alpha = 1$.

It is strictly increasing and continuous whenever $\alpha > 1$, by an analogue of Theorem 16.

Theorem 22. *For all α, β, and γ in* **ON**,

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma,$$
$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

Proof. Induction on γ in either case. □

Multiplication is not commutative, because for example for all n in ω, if $n > 1$, then

$$n \cdot \omega = \sup_{x<\omega}(n \cdot x) = \omega = \omega \cdot 1 < \omega \cdot n.$$

Theorem 23. *For all α and β in* **ON**,

$$\alpha \cdot \beta \cong \alpha \times \beta,$$

where $\alpha \times \beta$ is well-ordered by the rule,

$$(a, b) < (c, d) \Leftrightarrow b < d \lor (b = d \land a < c).$$

Proof. $\alpha \cdot \beta = \{\alpha \cdot y + x \colon x < \alpha \land y < \beta\}$, and the map

$$\alpha \cdot y + x \mapsto (x, y)$$

is a well-defined order-preserving bijection from $\alpha \cdot \beta$ to $\alpha \times \beta$. □

If $\gamma < \omega \cdot \omega$, then

$$\gamma = \omega \cdot b + a$$

for some a and b in ω. Then $\omega \cdot b + a$ is the **Cantor normal form** of γ. Compare with how every number less than $10 \cdot 10$ can be written as

$$10 \cdot b + a$$

for some a and b in 10 (which is $\{0, 1, 2, \ldots, 9\}$).

We add Cantor normal forms of elements of $\omega \cdot \omega$ by the rule (2) from Tuesday.

Theorem 24. *For each α in ω, the product $\alpha \cdot \omega$ is closed under addition.*

Proof. Suppose $\beta < \alpha \cdot \omega$. Then

$$\beta < \alpha \cdot n$$

for some n in ω, and thus

$$\beta \cdot 2 < \alpha \cdot n \cdot 2 < \alpha \cdot \omega. \qquad \square$$

Thus 0, ω, and $\omega \cdot \omega$ are closed under addition. Here $\omega \cdot \omega$ is ω^2 by the following definition:

1. $\alpha^0 = 1$.
2. $\alpha^{\beta'} = \alpha^\beta \cdot \alpha$.
3. $\lim(\gamma) \Rightarrow \alpha^\gamma = \sup\{\alpha^x : 0 < x < \gamma\}$.

We shall see that every nonzero element of ω^ω has a Cantor normal form

$$\omega^{n_0} \cdot k_0 + \omega^{n_1} \cdot k_1 + \cdots + \omega^{n_m} \cdot k_m,$$

where

$$\omega > n_0 > n_1 > \cdots > n_m,$$
$$0 < k_0 < \omega \wedge 0 < k_1 < \omega \wedge \cdots \wedge 0 < k_m < \omega.$$

The same will be true for *every* ordinal, if we remove the requirement $\omega > n_0$.

5 Friday

The following are now strictly increasing and continuous:

- $x \mapsto \alpha + x$ for all α,
- $x \mapsto \alpha \cdot x$ when $\alpha > 0$,
- $x \mapsto \alpha^x$ when $\alpha > 1$.

Also

$$\alpha^0 = 1,$$
$$\alpha^1 = \alpha^{0'} = \alpha^0 \cdot \alpha = 1 \cdot \alpha = \alpha,$$
$$\alpha^2 = \alpha^{1'} = \alpha^1 \cdot \alpha = \alpha \cdot \alpha.$$

Theorem 25. *For all* α *in* **ON**,

1. $0^\alpha = 0$ *if* $\alpha > 0$.
2. $1^\alpha = 1$.

Proof. Induction. □

Theorem 26. *For all* α, β, *and* γ *in* **ON**,

$$\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma,$$
$$\alpha^{\beta\cdot\gamma} = (\alpha^\beta)^\gamma.$$

Proof. Induction on γ. □

Yesterday we could assign to each element γ of ω^2 a Cantor normal form $\omega \cdot n + m$ as follows.

1. If $\gamma < \omega$, we are done. Suppose now

$$\omega \leqslant \gamma < \omega^2 = \omega \cdot \omega.$$

2. We let

$$n = \sup\{x \colon \omega \cdot x \leqslant \gamma\},$$

a nonzero element of ω, and then

$$\omega \cdot n \leqslant \gamma < \omega \cdot n' = \omega \cdot n + \omega.$$

3. We let

$$m = \sup\{x \colon \omega \cdot n + x \leqslant \gamma\},$$

an element of ω, and then

$$\gamma = \omega \cdot n + m.$$

To find the Cantor normal form of an arbitrary ordinal, we need to know that it is bounded by a power of ω. For this, we use the following.

Theorem 27. *For all α in* **ON**, *$x \mapsto x \cdot \alpha$ is increasing.*

Proof. Induction on α. □

Lemma 3. *If $\alpha \geqslant 2$, then for all γ in* **ON**,

$$\gamma \leqslant \alpha^\gamma.$$

Proof. Induction.

1. $0 < 1 = \alpha^0$.
2. Suppose $\beta < \alpha^\beta$ for some β.
   a) If $\beta = 0$, then $\beta' = 1 < \alpha = \alpha^{\beta'}$.
   b) If $\beta > 0$, then by Theorem 27,

$$\beta' = \beta + 1 \leqslant \beta + \beta = \beta \cdot 2 \leqslant \beta \cdot \alpha \leqslant \alpha^\beta \cdot \alpha = \alpha^{\beta'}.$$

3. Suppose β is a limit and the claim is true when $\gamma < \beta$. Then

$$\beta = \sup_{x<\beta} x \leqslant \sup_{x<\beta}(\alpha^x) = \alpha^\beta. \qquad \square$$

We can now find the Cantor normal form of an arbitrary nonzero γ in **ON** as follows. More generally, instead of ω as a base, we can use an arbitrary α, as long as $\alpha > 1$.

1. We know from Lemma 3 that

$$\gamma < \alpha^{\gamma'},$$

and therefore γ' is an upper bound of $\{x\colon \alpha^x \leqslant \gamma\}$, which is also nonempty, because it contains 0. We let

$$\beta_1 = \sup\{x\colon \alpha^x \leqslant \gamma\},$$

and then, by Theorem 18,

$$\begin{aligned}\alpha^{\beta_1} &= \alpha^{\sup\{x\colon \alpha^x \leqslant \gamma\}}\\ &= \sup\{\alpha^x\colon \alpha^x \leqslant \gamma\} \leqslant \gamma < \alpha^{\beta_1{}'} = \alpha^{\beta_1} \cdot \alpha.\end{aligned}$$

2. We let

$$\alpha_1 = \sup\{x\colon \alpha^{\beta_1} \cdot x \leqslant \gamma\},$$

a nonzero element of α, and then, again by Theorem 18,

$$\begin{aligned}\alpha^{\beta_1} \cdot \alpha_1 &= \alpha^{\beta_1} \cdot \sup\{x\colon \alpha^{\beta_1} \cdot x \leqslant \gamma\}\\ &= \sup\{\alpha^{\beta_1} \cdot x\colon \alpha^{\beta_1} \cdot x \leqslant \gamma\} \leqslant \gamma\\ &< \alpha^{\beta_1} \cdot \alpha_1{}' = \alpha^{\beta_1} \cdot \alpha_1 + \alpha^{\beta_1}.\end{aligned}$$

3. We let

$$\delta = \sup\{x\colon \alpha^{\beta_1} \cdot \alpha_1 + x \leqslant \gamma\},$$

an element of α^{β_1}, and then, again by Theorem 18,

$$\begin{aligned}\alpha^{\beta_1}\cdot\alpha_1+\delta &= \alpha^{\beta_1}\cdot\alpha_1+\sup\{x\colon \alpha^{\beta_1}\cdot\alpha_1+x\leqslant\gamma\}\\ &= \sup\{\alpha^{\beta_1}\cdot\alpha_1+x\colon \alpha^{\beta_1}\cdot\alpha_1+x\leqslant\gamma\}\leqslant\gamma\\ &\qquad < \alpha^{\beta_1}\cdot\alpha_1+\delta' = (\alpha^{\beta_1}\cdot\alpha_1+\delta)'.\end{aligned}$$

Thus

$$\gamma = \alpha^{\beta_1}\cdot\alpha_1+\delta, \qquad \delta<\alpha^{\beta_1}.$$

If $\delta>0$, then we apply to δ the same process that we did to γ. Ultimately we obtain, for some m in ω, the expansion

$$\gamma = \alpha^{\beta_1}\cdot\alpha_1+\alpha^{\beta_2}\cdot\alpha_2+\cdots+\alpha^{\beta_m}\cdot\alpha_m, \tag{3}$$

where

$$\beta_1>\beta_2>\cdots>\beta_m, \qquad \bigwedge_{1\leqslant j\leqslant m}\alpha>\alpha_j>0.$$

The expansion has finitely many terms, since **ON** is well-ordered, so the strictly decreasing sequence of β_j must end. If $m=0$, the expansion is 0.

We have shown

$$\alpha+\beta\cong\alpha\sqcup\beta, \qquad \alpha\cdot\beta\cong\alpha\times\beta.$$

Now let

$${}^{\beta}\alpha = \{\text{functions from } \beta \text{ to } \alpha\}.$$

Then we have established a function $f\mapsto f_\gamma$ from α^β to ${}^\beta\alpha$, where, if γ is as in (3), with $\beta>\beta_1$, then

$$f_\gamma(x) = \begin{cases}\alpha_j, & \text{if } x=\beta_j,\\ 0, & \text{otherwise.}\end{cases}$$

The function is injective, briefly because all of the functions mentioned at the beginning of today are strictly increasing. The range of the function is

$$\bigl\{f \in {}^{\beta}\alpha \colon \{x \in \beta \colon f(x) \neq 0\} \text{ is finite}\bigr\},$$

and we can well-order this so that it is isomorphic to α^{β}.

6 Saturday

We develop some rules for computing with Cantor normal forms. We have already effectively proved the following.

Lemma 4. *If* $\alpha \leqslant \beta$, *then the equation*

$$\alpha + x = \beta$$

is solved by $\sup\{x\colon \alpha + x \leqslant \beta\}$.

Proof. Letting $\gamma = \sup\{x\colon \alpha + x \leqslant \beta\}$, We compute

$$\begin{aligned}\alpha + \gamma &= \alpha + \sup\{x\colon \alpha + x \leqslant \beta\} \\ &= \sup\{\alpha + x\colon \alpha + x \leqslant \beta\} \leqslant \beta < \alpha + \gamma' = (\alpha + \gamma)',\end{aligned}$$

so $\alpha + \gamma = \beta$. □

Lemma 5. *If* $\alpha > 0$ *and* $n < \omega$, *then* $n + \omega^\alpha = \omega^\alpha$.

Proof. 1. We know the claim is true when $\alpha = 1$.

2. Hence, when $\beta \geqslant 1$,

$$n + \omega^{\beta'} \leqslant \omega^\beta + \omega^{\beta+1} = \omega^\beta \cdot (n + \omega) = \omega^{\beta'}.$$

(This did not require an inductive hypothesis.)

3. If β is a limit and the claim is true when $\alpha < \beta$, then (in the standard way)

$$n + \omega^\beta = n + \sup_{x<\beta} \omega^x = \sup_{x<\beta}(n + \omega^x) = \sup_{x<\beta} \omega^x = \omega^\beta. \quad \square$$

Theorem 28. *If* $\alpha < \beta$, *then*

$$\omega^\alpha + \omega^\beta = \omega^\beta.$$

Proof. By Lemma 4, for some γ, $\beta = \alpha + \gamma$, and then

$$\omega^\alpha + \omega^\beta = \omega^\alpha + \omega^{\alpha+\gamma} = \omega^\alpha \cdot (1 + \omega^\gamma) = \omega^\alpha \cdot \omega^\gamma = \omega^\beta. \quad \square$$

We have seen that, every α has a Cantor normal form, given by

$$\alpha = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_m} \cdot a_m,$$

where

$$\alpha_1 > \cdots > \alpha_m, \qquad \bigwedge_{1 \leqslant j \leqslant m} \omega > a_j > 0.$$

If $\alpha > 0$, so that $m \geqslant 1$, let us say

$$\deg(\alpha) = \alpha_1, \qquad \operatorname{lc}(\alpha) = a_1,$$

the **degree** and **leading coefficient** of α. Thus, for every positive ordinal α, for some ordinal β,

$$\alpha = \omega^{\deg(\alpha)} \cdot \operatorname{lc}(\alpha) + \beta, \qquad \deg(\beta) < \deg(\alpha).$$

By Theorem 28, if $\deg(\alpha) < \deg(\beta)$, then

$$\alpha + \beta = \beta.$$

Now we can compute sums of Cantor normal forms. Thus for example

$$(\omega^{\omega+5} \cdot 2 + \omega^3 \cdot 5 + 1) + (\omega^\omega + \omega^2 \cdot 6)$$
$$= \omega^{\omega+5} \cdot 2 + \omega^\omega + \omega^2 \cdot 6.$$

Lemma 6. *If* $\alpha > 0$ *and* $n \in \mathbb{N}$, *then* $n \cdot \omega^\alpha = \omega^\alpha$.

Proof. For some β, $\alpha = 1 + \beta$. We know $n \cdot \omega = \omega$, and then

$$n \cdot \omega^\alpha = n \cdot \omega \cdot \omega^\beta = \omega \cdot \omega^\beta = \omega^\alpha. \qquad \square$$

Theorem 29. *Let* $\beta > \deg(\alpha)$ *and* $n \in \mathbb{N}$.

1. *If* $k \in \mathbb{N}$, *then*

$$(\omega^\beta \cdot n + \alpha) \cdot k = \omega^\beta \cdot n \cdot k + \alpha.$$

2. *If* $\gamma \geqslant 1$, *then*

$$(\omega^\beta \cdot n + \alpha) \cdot \omega^\gamma = \omega^{\beta+\gamma}.$$

Proof. 1. We use (finite) induction.

a) The claim is trivially true when $k = 1$.

b) If it is true when $k = \ell$, then

$$\begin{aligned}(\omega^\beta \cdot n + \alpha) \cdot (\ell + 1) &= (\omega^\beta \cdot n + \alpha) \cdot \ell + \omega^\beta \cdot n + \alpha \\ &= \omega^\beta \cdot n \cdot \ell + \alpha + \omega^\beta \cdot n + \alpha \\ &= \omega^\beta \cdot n \cdot \ell + \omega^\beta \cdot n + \alpha \\ &= \omega^\beta \cdot n \cdot (\ell + 1) + \alpha.\end{aligned}$$

2. As in the proof of Lemma 6, we need only prove the claim when $\gamma = 1$. We have

$$\begin{aligned}\omega^{\beta+1} = \sup_{x<\omega}(\omega^\beta \cdot x) &\leqslant \sup_{x<\omega}((\omega^\beta \cdot n + \alpha) \cdot x) \\ &\leqslant \sup_{x<\omega}(\omega^\beta \cdot (n+1) \cdot x) = \omega^{\beta+1},\end{aligned}$$

while also $\sup_{x<\omega}((\omega^\beta \cdot n + \alpha) \cdot x) = (\omega^\beta \cdot n + \alpha) \cdot \omega$. $\square$

Now we can compute products of Cantor normal forms; for, we have (under the obvious conditions)

$$\begin{aligned}&(\omega^\beta \cdot n + \alpha) \cdot (\omega^{\gamma_1} \cdot d_1 + \cdots + \omega^{\gamma_m} \cdot d_m + d_{m+1}) \\ &\qquad = \omega^{\beta+\gamma_1} \cdot d_1 + \cdots + \omega^{\beta+\gamma_m} \cdot d_m + \omega^\beta \cdot n \cdot d_{m+1} + \alpha,\end{aligned}$$

the last two terms being deleted when $d_{m+1} = 0$.

Theorem 30. *If $k \in \mathbb{N}$, $n \in \omega$, and $\omega \leqslant \alpha$, then*

$$k^{\omega^{n+1}} = k^{\omega^{1+n}} = (k^{\omega})^{\omega^n} = \omega^{\omega^n},$$
$$k^{\omega^\alpha} = k^{\omega^{1+\alpha}} = \omega^{\omega^\alpha}.$$

Lemma 7. *α is a non-successor if and only if, for some β, $\alpha = \omega \cdot \beta$.*

Proof. 0 is not a successor and is $\omega \cdot 0$. Suppose $\alpha > 0$. In Cantor normal form,

$$\alpha = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_m} \cdot a_m + a_{m+1},$$

where $\alpha_m > 0$ and $a_{m+1} \in \omega$. By Lemma 4, for some β_i,

$$\alpha = \omega \cdot (\omega^{\beta_1} \cdot a_1 + \cdots + \omega^{\beta_m} \cdot a_m) + a_{m+1},$$

which is a limit if and only if $a_{m+1} > 0$. □

Theorem 31. *If γ is a limit, then*

$$(\omega^\beta \cdot n + \alpha)^\gamma = \omega^{\beta \cdot \gamma}.$$

We end with **cardinalities.** If there is, from A to B,
- an injection, $A \preccurlyeq B$;
- a bijection, $A \approx B$;

is written.

Theorem 32 (Schröder–Bernstein). *If*

$$f\colon A \xrightarrow{\preccurlyeq} B, \qquad\qquad g\colon B \xrightarrow{\preccurlyeq} A,$$

then

$$A \approx B.$$

Zermelo's proof. Under the hypothesis,

$$A \approx g \circ f[A], \qquad g \circ f[A] \subseteq g[B] \subseteq A, \qquad g[B] \approx B.$$

Thus, assuming

$$A \subseteq B \subseteq C, \qquad\qquad f\colon C \xrightarrow{\approx} A,$$

we find a bijection g from B onto A. Let

$$\mathscr{E} = \{X \subseteq B\colon (B \smallsetminus A) \cup f[X] \subseteq X\}.$$

Then

$$B \in \mathscr{E}, \qquad\qquad X \in \mathscr{E} \Rightarrow (B \smallsetminus A) \cup F[X] \in \mathscr{E}.$$

Let

$$D = \bigcap \mathscr{E} = \{t\colon \forall X\ (X \in \mathscr{E} \Rightarrow t \in X)\}.$$

Then

$$D \in \mathscr{E}, \qquad\qquad D = (A \smallsetminus B) \cup f[D]. \qquad\qquad \square$$

Theorem 33. *If* $\alpha \cdot \beta \neq 0$ *and* $\max(\alpha, \beta) \geqslant \omega$, *then*

$$\alpha + \beta \approx \max(\alpha, \beta), \qquad\qquad \alpha \cdot \beta \approx \max(\alpha, \beta).$$

Proof. Since

$$\alpha + \beta \approx \alpha \sqcup \beta \approx \beta \sqcup \alpha \approx \beta + \alpha,$$
$$\alpha \cdot \beta \approx \alpha \times \beta \approx \beta \times \alpha \approx \beta \cdot \alpha,$$

and also by Lemmas 5 and 6 we may assume $\omega \leqslant \alpha \leqslant \beta$. Then

$$\beta \leqslant \alpha + \beta \leqslant \beta + \beta = \beta \cdot 2 < \beta \cdot \alpha \leqslant \beta \cdot \beta = \beta^2.$$

Thus it is enough to show

$$\beta \approx \beta^2.$$

This is true when $\beta = \omega$, since for example if

$$t_n = \frac{n \cdot (n+1)}{2},$$

then $(k, m) \mapsto t_{k+m} + k$ is a bijection from $\omega \times \omega$ to ω. Then by finite induction, for all n in $\mathbb{N}$,

$$\omega^n \approx \omega.$$

Letting $\gamma = \deg(\beta)$, we know for some δ, where $\deg(\delta) < \gamma$,

$$\beta = \omega^\gamma \cdot \mathrm{lc}(\beta) + \delta \approx \delta + \mathrm{lc}(\beta) \cdot \omega^\gamma = \omega^\gamma.$$

We may assume now $\gamma \geqslant \omega$. Since for any α and β

$$\alpha^\beta \approx \{f \in {}^\beta\alpha \colon \operatorname{card}(\operatorname{supp}(f)) < \omega\}, \tag{4}$$

where by definition

$$\operatorname{card}(\operatorname{supp}(f)) < \omega \Leftrightarrow \{x \in \beta \colon f(x) \neq 0\} \text{ is finite},$$

we have generally

$$\alpha \approx \gamma \wedge \beta \approx \delta \Rightarrow \alpha^\beta \approx \gamma^\delta.$$

In our case,

$$\beta^2 \approx (\omega^\gamma)^2 \approx \omega^{\omega^{\deg(\gamma)} \cdot 2} \approx \omega^{2 \cdot \omega^{\deg(\gamma)}} = \omega^{\omega^{\deg(\gamma)}} \approx \omega^\gamma \approx \beta. \quad \square$$

Corollary. *For all infinite α, for all n in $\mathbb{N}$,*

$$\alpha^n \approx \alpha.$$

Theorem 34. *If* $\min(\alpha,\beta) \leqslant 2$ *and* $\omega \leqslant \max(\alpha,\beta)$, *then*

$$\alpha^\beta \approx \max(\alpha,\beta).$$

Proof. From Lemma 3 we know

$$\max(\alpha,\beta) \leqslant \alpha^\beta.$$

From (4), an element of α^β uniquely determines an ordered pair consisting of, for some n in ω,

- a subset $\{\beta_i \colon i < n\}$ of β, where $\beta_0 < \cdots < \beta_{n-1}$;
- a function $(\alpha_i \colon i < n)$ in ${}^n\alpha$.

Moreover, the finite subsets of β correspond precisely to the elements of 2^β. Thus

$$\alpha^\beta \preccurlyeq 2^\beta \times \alpha^\omega.$$

Then also

$$2^\beta \preccurlyeq \bigcup_{x<\omega} {}^x\beta \preccurlyeq \bigsqcup_{x<\omega} \beta \approx \omega \times \beta \approx \beta,$$

and likewise

$$\alpha^\omega \approx \bigcup_{x<\omega} {}^x\alpha \approx \alpha. \qquad \square$$