The Cantor Set

A winter course at the Nesin Matematik Köyü

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Abstract

We show how the Cantor set, with the order topology induced from $\mathbb{R}$, is homeomorphic to the power set of the natural numbers, as equipped with the Tychonoff topology. We obtain an analogy between the Heine–Borel Theorem and the Tychonoff Theorem—and with the Compactness Theorem of model theory.

Preface

These notes are based on a course of seven lectures, 2×50 minutes each. I have thoroughly revised the notes from a similar course in January 2017. Those notes were divided by day; the present notes are divided by topic and may differ more from what actually happened in class. The Summary on page 4 lists the numbered theorems of these notes (other results are called lemmas and have their own numbering); what I did day by day is roughly as follows.

Monday Theorem 1.
Tuesday Theorem 2, Cantor's Theorem, Russell Paradox.
Wednesday Schröder–Bernstein, Cantor Intersection.
Thursday Theorems 7, 8, and 9.
Friday Theorem 10, Heine–Borel.
Saturday König's Lemma, Tychonoff Theorem.
Sunday Compactness Theorem, Stone Representation.

The section on topological rings is an afterthought that connects the course to the following week's course, on finite fields. Two students attended at least parts of both courses. For the Cantor set course, I had thirty or forty students the first day; down to about fifteen on Thursday, six on Friday, and four on Sunday.

Summary

The Cantor Set is the intersection C of an infinite family of sets, as in (1.5). The family forms a decreasing chain, as in (1.1). Each set in the family is a union, as in (2.5), of finitely many sets, which closed intervals of $\mathbb{R}$, as in (2.6).

1. C is the set of points in $[0, 1]$ that can be written in base 3 without the digit 1 (page 14).
2. C is equipollent with the set $\mathscr{P}(\omega)$ of subsets of the set ω of natural numbers (page 16).
3. $\mathscr{P}(\omega)$ is uncountable, By Cantor's Theorem (page 17).
4. The proof is as for the Russell Paradox (page 19).
5. $\mathscr{P}(\omega)$ and $\mathbb{R}$ are equipollent, by the Cantor–Schröder–Bernstein Theorem (page 23).
6. As the intersection of a decreasing chain of nonempty bounded closed sets, C is nonempty, by the Cantor Intersection Theorem (page 26).
7. $\mathbb{R}$ is a topological space with open intervals as a basis (page 27).
8. C is closed in this topology (page 28).
9. $\mathbb{R}$ has only itself and the empty set as clopen subsets (page 29).
10. C has many clopen subsets, namely its intersection with the intervals mentioned above (page 30).
11. C is compact by the Heine–Borel Theorem (page 32), so every clopen subset is a finite union of the clopen subsets just mentioned.

12. Our proof of the Heine–Borel Theorem is analous to that of König's Lemma (page 36).
13. —Also to that of the Tychonoff Theorem (page 39).
14. —And of the Compactness Theorem for propositional logic (page 46).
15. This Compactness Theorem is a special case of the Stone Representation Theorem (page 52), though the Compactness Theorem of first-order logic is not.

With the topology induced from $\mathbb{R}$, the Cantor set is a topological ring in two different ways, as (isomorphic to) the Boolean ring $\mathscr{P}(\omega)$ and the ring $\mathbb{Z}_{(2)}$ of dyadic integers.

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1 Cardinality

1.1 The Cantor Set

The *Cantor set* is a certain subset of $\mathbb{R}$. For the sake of defining it, we let

$$\begin{aligned} C_0 &= [0, 1], \\ C_1 &= [0, 1/3] \cup [2/3, 1], \\ C_2 &= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \end{aligned}$$

and so on, as in Figure 1.1. Thus each C_n is a union of bounded closed intervals, and we obtain C_{n+1} from C_n by removing the middle third of each component interval. Presently we shall find a "closed" expression for each set C_n, ultimately as in (1.6). For now, we make three observations.

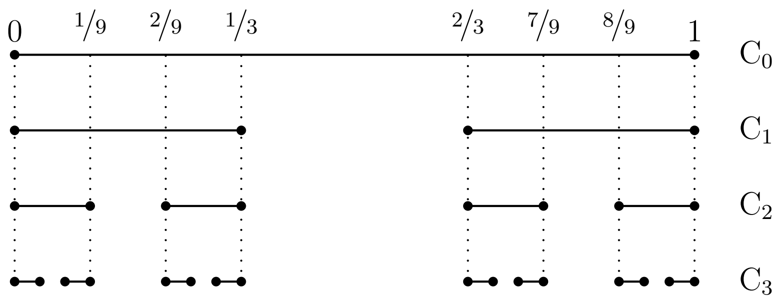

Figure 1.1: Some sets C_n

1. The sets C_n compose a **decreasing chain:**

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \tag{1.1}$$

2. Each C_n is the union of 2^n-many disjoint bounded closed intervals.
3. Each of those intervals has length 3^{-n}.

Thus the **measure** of C_n is $(2/3)^n$; we may write

$$\mu(C_n) = \left(\frac{2}{3}\right)^n,$$

where μ is the minuscule Greek letter mu. By definition, the **Cantor Set** is the subset $\boxed{C}$ of $\mathbb{R}$ given by

$$C = C_0 \cap C_1 \cap C_2 \cap \cdots$$

of $\mathbb{R}$. We may write the intersection here as

$$\bigcap_{n=0}^{\infty} C_n.$$

If this has a measure, it ought to be $\lim_{n\to\infty}(2/3)^n$, which is 0. Nonetheless, C is not empty. This will turn out to be a special case of the Cantor Intersection Theorem, page 26. For example, C contains the endpoints of the component intervals of each C_n. To specify these points, we observe

$$\begin{aligned} C_0 &= \left[0, \frac{1}{3^0}\right], \\ C_1 &= \left[0, \frac{1}{3^1}\right] \cup \left[\frac{2}{3^1}, \frac{2}{3^1} + \frac{1}{3^1}\right], \end{aligned} \tag{1.2}$$

and

$$\mathrm{C}_2 = \left[0, \frac{1}{3^2}\right] \cup \left[\frac{2}{3^2}, \frac{2}{3^2} + \frac{1}{3^2}\right] \\ \cup \left[\frac{2}{3^1}, \frac{2}{3^1} + \frac{1}{3^2}\right] \cup \left[\frac{2}{3^1} + \frac{2}{3^2}, \frac{2}{3^1} + \frac{2}{3^2} + \frac{1}{3^2}\right]. \tag{1.3}$$

In general, each C_n is the union of all of the intervals of the form

$$\left[\frac{d_1}{3^1} + \dots + \frac{d_n}{3^n}, \frac{d_1}{3^1} + \dots + \frac{d_n}{3^n} + \frac{1}{3^n}\right],$$

where each d_k is either 0 or 2. In Sigma-notation, the intervals are

$$\left[\sum_{k=1}^{n} \frac{d_k}{3^k}, \sum_{k=1}^{n} \frac{d_k}{3^k} + \frac{1}{3^n}\right].$$

1.2 Natural numbers

In seeking to understand the Cantor set C, we shall make use, at least notationally, of the natural numbers according to the definition of von Neumann. We shall denote the *set* of natural numbers by $\boxed{\omega,}$ which is the minuscule Greek letter omega. Informally,

$$\omega = \{0, 1, 2, \dots\}.$$

We make a recursive definition as follows.

1. 0 is $\varnothing$, the empty set, and this is in ω.
2. If n is in ω, then the **successor** of n, namely $n + 1$, is $n \cup \{n\}$, and this is in ω.
3. Nothing else is in ω.

It is a set-theoretical challenge to establish that ω is a well-defined set; we accept that it is.* We have

$$0 = \varnothing, \qquad 1 = \{0\}, \qquad 2 = \{0, 1\},$$

and for all n in ω,

$$n = \{0, \dots, n-1\} = \{x \in \omega \colon x < n\}. \tag{1.4}$$

Here the ordering $<$ of ω is $\subset$, and one can show that this is also $\in$. Easily ω is **well-ordered,** that is, each nonempty subset has a least element; for the least element of a nonempty subset A of ω is the intersection $\bigcap A$.

We shall denote the set of functions from a set B to a set A by

$$^{B}A.$$

Putting everything together, we have

$$^{n}2 = \bigl\{\text{functions from } \{0, \dots, n-1\} \text{ to } \{0, 1\}\bigr\}.$$

A typical element of $^{n}2$ can be written as either of

$$(e_0, \dots, e_{n-1}), \qquad \boldsymbol{e}$$

*Each element of ω *includes,* as a subset, each of its elements; it is also well-ordered, in the sense to be defined presently; therefore, by definition, each element of ω is an **ordinal.** There may be other ordinals, such as ω itself. Still, ω would be a **limit** ordinal, in the sense of being neither 0 nor a successor. By definition, ω is the *class,* in the sense of page 18, of ordinals that neither *are* limits nor *contain* limits. Though it is not usually expressed this way, the *Axiom of Infinity* is that the class ω is a set. In this way, the Axiom is parallel to most of the other Zermelo–Fraenkel Axioms, namely those whereby certain classes are sets.

—in handwriting, $\boxed{\vec{e}}$—, where each e_k is in 2. We can now say precisely

$$\mathrm{C} = \bigcap_{n\in\omega} \mathrm{C}_n, \tag{1.5}$$

where

$$\mathrm{C}_n = \bigcup_{e\in{}^n2} \left[\sum_{k\in n} \frac{2e_k}{3^{k+1}}, \sum_{k\in n} \frac{2e_k}{3^{k+1}} + \frac{1}{3^n}\right]. \tag{1.6}$$

In **ternary** (base-3) notation,

$$\sum_{k\in n} \frac{2e_k}{3^{k+1}} = 0.d_1 \cdots d_n,$$

where $d_{k+1} = 2e_k$. In this notation then, from (1.2) and (1.3) we have

$$\begin{gathered}
\mathrm{C}_0 = [0,1], \\
\mathrm{C}_1 = [0,0.1] \cup [0.2,1], \\
\mathrm{C}_2 = [0,0.01] \cup [0.02,0.1] \cup [0.2,0.21] \cup [0.22,1],
\end{gathered}$$

and likewise

$$\begin{aligned}
\mathrm{C}_3 = {} & [0,0.001] \cup [0.002,0.01] \cup [0.02,0.021] \cup [0.022,0.1] \\
& \cup [0.2,0.201] \cup [0.202,0.21] \cup [0.22,0.221] \cup [0.222,1].
\end{aligned}$$

Again, C contains all of the endpoints here. Some of these are shown in Figure 1.2. Since

$$\sum_{n\in\omega} \frac{2}{3^{n+1}} = 1,$$

we can replace terminal digits 1 with 0 followed by repeating 2:

$$0.d_1 \cdots d_n 1 = 0.d_1 \cdots d_n 0222 \cdots = 0.d_1 \cdots d_n 0\overline{2}.$$

$$
\begin{array}{rl}
\omega & 1 \\
\{0,1,2\} & 0.222 \\
\omega \setminus \{2\} & 0.221 \\
\{0,1\} & 0.22 \\
\\
\omega \setminus \{1\} & 0.21 \\
\{0,2\} & 0.202 \\
\omega \setminus \{1,2\} & 0.201 \\
\{0\} & 0.2 \\
\\
X & \displaystyle\sum_{k \in X} \frac{2}{3^{k+1}} \\
\\
\omega \setminus \{0\} & 0.1 \\
\{1,2\} & 0.022 \\
\omega \setminus \{0,2\} & 0.021 \\
\{1\} & 0.02 \\
\\
\omega \setminus \{0,1\} & 0.01 \\
\{2\} & 0.002 \\
\omega \setminus \{0,1,2\} & 0.001 \\
\varnothing & 0
\end{array}
$$

Figure 1.2: The Cantor set

Thus each of the endpoints of a component interval of C_n is a series

$$\sum_{k=0}^{\infty} \frac{2e_k}{3^{k+1}},$$

where each e_k is in the set 2. Then the series is just

$$\sum_{k \in A} \frac{2}{3^{k+1}},$$

where A is the subset $\{k \in \omega : e_k = 1\}$ of ω.

The set of subsets any set A is the called the **power set** of A, for reasons to be suggested by (1.14). We shall denote the power set of A by $\boxed{\mathscr{P}(A).}$ Thus

$$\mathscr{P}(A) = \{X : X \subseteq A\} = \{X : \forall y\ (y \in X \implies y \in A)\}. \tag{1.7}$$

1 Theorem. *The Cantor set consists precisely of the points of the interval* $[0, 1]$ *that can be written in ternary notation without use of the digit* 1. *That is,*

$$\mathrm{C} = \left\{ \sum_{k \in X} \frac{2}{3^{k+1}} : X \in \mathscr{P}(\omega) \right\}.$$

Proof. Using (1.1) and (1.5), and applying an infinitary De Morgan law, we compute

$$\begin{aligned}[0,1] \smallsetminus \mathrm{C} = \mathrm{C}_0 \smallsetminus \bigcap_{n \in \omega} \mathrm{C}_{n+1} &= \bigcup_{n \in \omega} (\mathrm{C}_0 \smallsetminus \mathrm{C}_{n+1}) \\ &= \bigcup_{n \in \omega} (\mathrm{C}_n \smallsetminus \mathrm{C}_{n+1}). \end{aligned} \tag{1.8}$$

In ternary notation, by (1.6),

$$\mathrm{C}_0 \smallsetminus \mathrm{C}_1 = (0.1, 0.2),$$

which is just the set of numbers in $[0, 1]$ that need the digit 1 in the first place after the point. Likewise,

$$\mathrm{C}_1 \smallsetminus \mathrm{C}_2 = (0.01, 0.02) \cup (0.21, 0.22),$$

consisting of numbers needing the digit 1 not in the first place, but in the second place after the point. In general,

$$\mathrm{C}_n \smallsetminus \mathrm{C}_{n+1} = \bigcup_{\boldsymbol{e} \in {}^n 2} \left(\sum_{k \in n} \frac{2e_k}{3^{k+1}} + \frac{1}{3^{n+1}}, \sum_{k \in n} \frac{2e_k}{3^{k+1}} + \frac{2}{3^{n+1}} \right),$$

and so $[0, 1] \smallsetminus \mathrm{C}$ consists of the points of $[0, 1]$ that do need 1 at any place in their ternary expansions. □

We now define a function g from $\mathscr{P}(\omega)$ to $\mathbb{R}$ by

$$\mathrm{g}(A) = \sum_{k \in A} \frac{2}{3^{k+1}}. \tag{1.9}$$

Then

$$\mathrm{C} = \mathrm{g}[\mathscr{P}(\omega)] = \{\mathrm{g}(X) \colon X \in \mathscr{P}(\omega)\},$$

which is the **range** of g or the **image** of $\mathscr{P}(\omega)$ under g. Note that g is increasing, in the sense that

$$A \subset B \implies \mathrm{g}(A) < \mathrm{g}(B). \tag{1.10}$$

Also g is additive in the sense that

$$A \cap B = \varnothing \implies \mathrm{g}(A \cup B) = \mathrm{g}(A) + \mathrm{g}(B). \tag{1.11}$$

Here we use the result from analysis that the sum of an absolutely convergent series is independent of the order of the terms.

The symbol $\boxed{\triangle}$ denotes the operation of taking the **symmetric difference** of two sets, so that

$$A \mathbin{\triangle} B = (A \smallsetminus B) \cup (B \smallsetminus A).$$

Then

$$A = B \iff A \mathbin{\triangle} B = \varnothing. \tag{1.12}$$

2 Theorem. *The function* g *given by* (1.9) *is injective.*

Proof. Suppose A and B are distinct subsets of ω. Then $A \mathbin{\triangle} B \neq \varnothing$ by (1.12), and so, since ω is well-ordered, we may let

$$m = \min(A \mathbin{\triangle} B).$$

Then

$$A \cap m = \{x \in A \colon x < m\} = \{x \in B \colon x < m\} = B \cap m.$$

We may assume $m \in B \smallsetminus A$. Then

$$\begin{gathered} A \subseteq (A \cap m) \cup \{x \in \omega \colon x > m\}, \\ (A \cap m) \cup \{m\} \subseteq B, \end{gathered}$$

and so, by (1.10) and (1.11),

$$\mathrm{g}(A) \leqslant \mathrm{g}(A \cap m) + \frac{1}{3^{m+1}} < \mathrm{g}(A \cap m) + \frac{2}{3^{m+1}} \leqslant \mathrm{g}(B).$$

In short, if $A \neq B$, then $\mathrm{g}(A) \neq \mathrm{g}(B)$. Thus g is injective. □

1.3 Comparison of sets

If there is a bijection from a set A to a set B, we shall write

$$A \approx B,$$

and we shall say that A and B are **equipollent** (words like "equipotent" are also used). By Theorem 2 then,

$$\mathscr{P}(\omega) \approx \mathrm{C}. \tag{1.13}$$

A set is called **countable** if it is equipollent with a subset of ω (possibly ω itself). We shall show presently that C is not countable.

If there is an injection from A to B, that is, an embedding of A in B, we may write simply

$$A \preccurlyeq B.$$

By Theorem 2 then,

$$\mathscr{P}(\omega) \preccurlyeq \mathbb{R}.$$

By definition

$$A \prec B \iff A \preccurlyeq B \;\;\&\;\; A \not\approx B.$$

3 Cantor's Theorem. *For all sets A,*

$$A \prec \mathscr{P}(A).$$

Proof. The map $x \mapsto \{x\}$ embeds A in $\mathscr{P}(A)$. Suppose some function f embeds A in $\mathscr{P}(A)$. Let

$$B = \{x \in A \colon x \notin f(x)\}.$$

Then $B \in \mathscr{P}(A)$, and for all c in A we have

$$c \in B \iff c \notin f(c).$$

Thus $B \neq f(c)$. This means f is not surjective onto $\mathscr{P}(A)$. □

By (1.13) now,

$$\omega \prec \mathrm{C},$$

and so C is uncountable. Note also that, for any set A,

$$\mathscr{P}(A) \approx {}^{A}2, \tag{1.14}$$

since there is a bijection $X \mapsto \chi_X$ from $\mathscr{P}(A)$ to ${}^{A}2$ given by

$$\chi_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \in A \smallsetminus B, \end{cases} \tag{1.15}$$

where B ranges over $\mathscr{P}(A)$. Here χ is the minuscule Greek letter khi or chi, standing for *characteristic,* since χ_B can be called the **characteristic function** of B; strictly this function depends on A as well. The inverse of $X \mapsto \chi_X$ is $\sigma \mapsto \sigma^{-1}(1)$, where, if $i \in 2$,

$$\sigma^{-1}(i) = \{k \in \omega \colon \sigma(k) = i\}. \tag{1.16}$$

By Cantor's Theorem,

$$A \prec {}^{A}2.$$

When $n \in \omega$, the size of the set ${}^{n}2$ is just 2^n. This, with (1.14), is why the power set is so called. Cantor's Theorem is a generalization of the theorem that $n < 2^n$.

The proof of Cantor's Theorem is similar to one proof that there are *classes* that are not sets. In set theory, a **class** is the collection

$$\{x \colon \varphi(x)\}$$

of sets that satisfy a formula $\varphi(x)$ of the language of set theory. We shall not define formulas now, but shall only give examples.

- For each set A,

$$A = \{x\colon x \in A\},$$

  the class of elements of A. Variables stand for sets; thus the language of set theory requires all elements of sets to be sets. Von Neumann's definition of the natural numbers conforms to this convention: numbers are sets, and their elements (which are other numbers) are sets.
- We let

$$\mathbf{V} = \{x\colon x = x\},$$

  the **universal class,** consisting of all of the sets.
- For an arbitrary class $\boldsymbol{D}$, generalizing (1.7), we let

$$\mathscr{P}(\boldsymbol{D}) = \{x\colon x \subseteq \boldsymbol{D}\},$$

  the class of all subsets of $\boldsymbol{D}$.

We now have

$$\mathscr{P}(\mathbf{V}) = \mathbf{V}. \tag{1.17}$$

1 **Exercise.** Why does (1.17) not contradict Cantor's Theorem?

4 **Russell Paradox.** *The class $\{x\colon x \notin x\}$ of sets that are not elements of themselves cannot be a set.*

Proof. If we denote the class by $\boldsymbol{R}$, then for every set b we have

$$b \in \boldsymbol{R} \iff b \notin b,$$

and so $b \neq \boldsymbol{R}$. Thus $\boldsymbol{R}$ is not a set. □

The conclusion of the Russell Paradox is that the class called $\boldsymbol{R}$ in the proof is a **proper class:** a class that is not a set.

The **Separation Axiom** is that every sub-class of a set is a set. Since every class is a sub-class of the universal class $\mathbf{V}$, it follows that this too is a proper class. But we already knew this from (1.17) and Cantor's Theorem.

We now define a function f, like g in (1.9), from $\mathscr{P}(\omega)$ to $\mathbb{R}$. We let

$$f(A)=\sum_{k\in A}\frac{1}{2^{k+1}}.$$

Then $f(\varnothing)=0$ and $f(\omega)=1$, and in general

$$f(A)\in[0,1].$$

Indeed,

$$f[\mathscr{P}(\omega)]=[0,1],$$

since every number in the interval has a binary expansion. However, some of these numbers have more than one binary expansion. For example, the equation $f(X)={}^1\!/\!{}_2$ has two solutions, $\{0\}$ and $\omega\setminus\{0\}$; so f is not injective. By the **Axiom of Choice,** f has a *right inverse* h, which means

$$f\bigl(h(a)\bigr)=a \tag{1.18}$$

for all y in $[0,1]$. Then h is injective, so

$$[0,1]\preccurlyeq\mathscr{P}(\omega). \tag{1.19}$$

In fact we can make an explicit definition of h once for all, without needing the Axiom of Choice. If $a\in[0,1]$, we define

$$h(a)=\{k\in\omega\colon a_k=1\},$$

where the a_k are defined recursively by

$$a_k=\begin{cases}0, & \text{if } a<\displaystyle\sum_{i\in k}\frac{a_i}{2^{i+1}}+\frac{1}{2^{k+1}},\\[2ex] 1, & \text{if } a\geqslant\displaystyle\sum_{i\in k}\frac{a_i}{2^{i+1}}+\frac{1}{2^{k+1}}.\end{cases}$$

This ensures (1.18). We have defined the a_n so that, for all k in ω, for some n in ω, we have $k \leqslant n$ and $a_n = 0$, unless $a = 1$.

We have

$$\mathbb{R} \approx (0, 1) \tag{1.20}$$

because the function

$$x \mapsto \frac{x - 1/2}{x \cdot (1 - x)},$$

restricted to $(0, 1)$, is a bijection onto $\mathbb{R}$, as in Figure 1.3. Indeed, the inverse can be computed as

$$y \mapsto \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{1 + y^2} - 1}{y}$$

away from 0. This is by the computations

$$\begin{gathered} y = \frac{x - 1/2}{x \cdot (1 - x)}, \\ yx - yx^2 = x - 1/2, \\ yx^2 + (1 - y)x - 1/2 = 0, \\ x = \frac{y - 1 \pm \sqrt{(1 - y)^2 + 2y}}{2y} = \frac{y - 1 \pm \sqrt{1 + y^2}}{2y}, \end{gathered}$$

and we let the ambiguous sign $\pm$ be $+$ to put x in the interval $(0, 1)$. In sum, by (1.13), (1.20), and (1.19),

$$\mathscr{P}(\omega) \approx \mathrm{C} \subseteq \mathbb{R} \approx (0, 1) \subseteq [0, 1] \preccurlyeq \mathscr{P}(\omega). \tag{1.21}$$

By the next theorem, we shall be able to conclude

$$\mathbb{R} \approx \mathscr{P}(\omega). \tag{1.22}$$

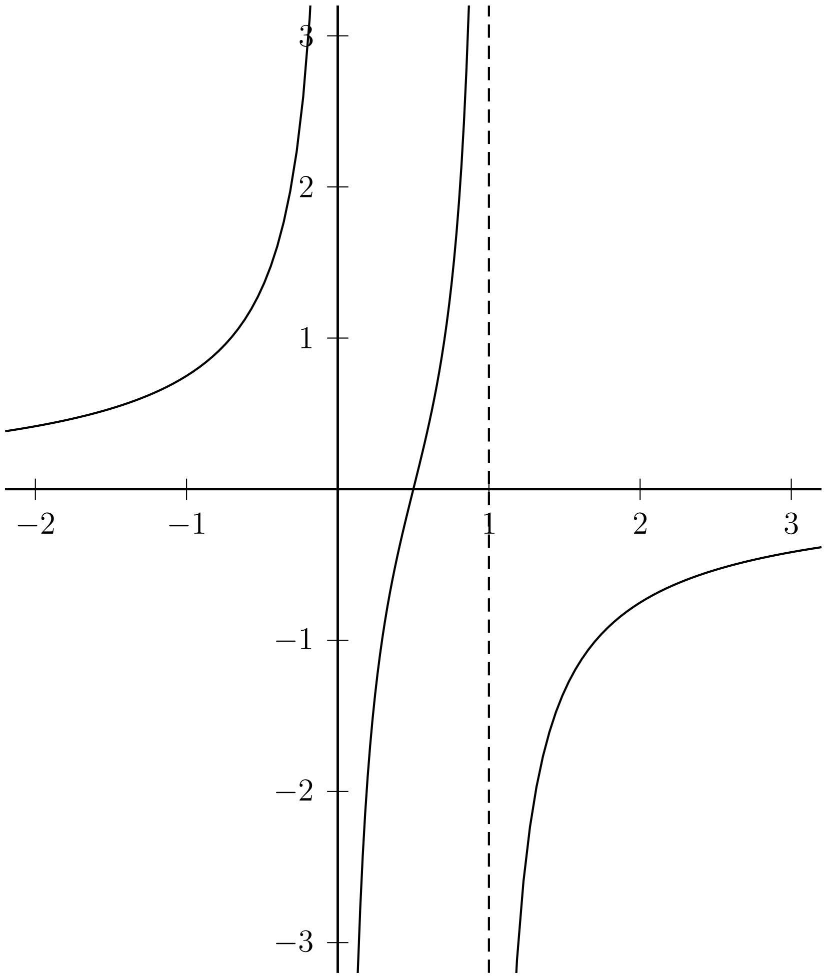

Figure 1.3: Graph of $y = \dfrac{x - {}^1\!/_2}{x \cdot (1 - x)}$

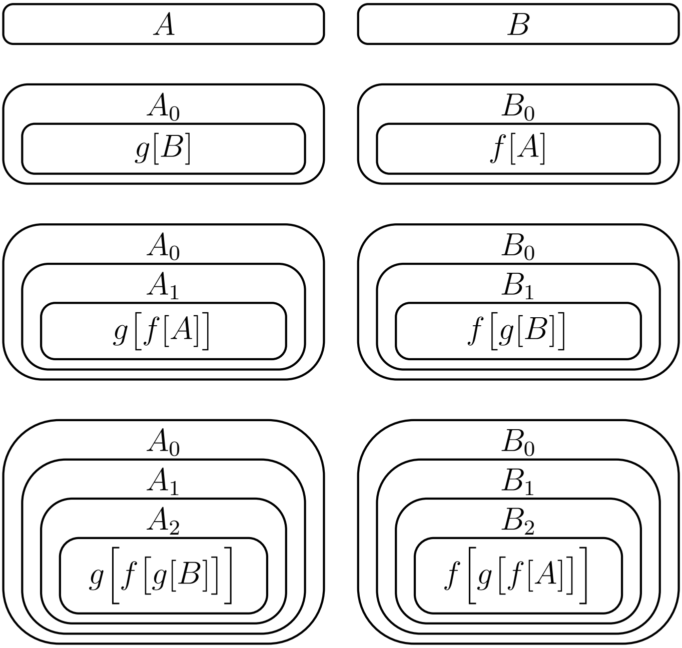

Figure 1.4: Cantor–Schröder–Bernstein Theorem

5 Cantor–Schröder–Bernstein Theorem. *If* $A \preccurlyeq B$ *and* $B \preccurlyeq A$, *then*

$$A \approx B.$$

Proof. Suppose f is an injection from A to B; and g, from B to A. By recursion, we define

$$A_0 = A \smallsetminus g[B], \qquad B_0 = B \smallsetminus f[A],$$
$$A_{n+1} = g[B_n], \qquad B_{n+1} = f[A_n].$$

See Figure 1.4. By induction, we prove

$$A_n \subseteq A \smallsetminus \bigcup_{k<n} A_k, \qquad B_n \subseteq B \smallsetminus \bigcup_{k<n} B_k.$$

This is clear when $n = 0$, and if true when $n = m$, then for example

$$B_{m+1} \subseteq f[A] \smallsetminus \bigcup_{k<m} B_{k+1},$$

but the latter is $B \smallsetminus \bigcup_{k<m+1} B_k$, since

$$f[A] = B \smallsetminus B_0. \tag{1.23}$$

Then also, not by induction,

$$A_{2n} \cup A_{2n+1} \approx B_{2n+1} \cup B_{2n},$$

and therefore

$$\bigcup_{n\in\omega} A_n \approx \bigcup_{n\in\omega} B_n.$$

Finally,

$$A \smallsetminus \bigcup_{n\in\omega} A_n \approx B \smallsetminus \bigcup_{n\in\omega} B_n,$$

since

$$\begin{aligned} f\left[A \smallsetminus \bigcup_{n\in\omega} A_n\right] &= f\left[A \smallsetminus \bigcup_{n\in\omega}\bigcup_{k<n} A_n\right] \\ &= f\left[\bigcap_{n\in\omega}\left(A \smallsetminus \bigcup_{k<n} A_n\right)\right] = \bigcap_{n\in\omega} f\left[A \smallsetminus \bigcup_{k<n} A_k\right] \\ &= \bigcap_{n\in\omega}\left(B \smallsetminus \bigcup_{k\leqslant n} B_k\right) = B \smallsetminus \bigcup_{n\in\omega} B_n, \end{aligned}$$

by an infinitary De Morgan law as in (1.8), and (1.23). □

2 Topology

2.1 Open and closed sets

We shall study how to generalize the following into the Cantor Intersection Theorem.

1 **Lemma.** *If a sequence* $(F_n : n \in \omega)$ *of closed, bounded intervals compose a decreasing chain, then*

$$\bigcap_{n\in\omega} F_n \neq \varnothing.$$

Proof. We can write each F_n as $[a_n, b_n]$. Then the sequence $(a_n : n \in \omega)$ is bounded and increasing, so it converges. Let

$$\lim_{n\to\infty} a_n = c. \tag{2.1}$$

Since

$$a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant c \leqslant \cdots \leqslant b_2 \leqslant b_1 \leqslant b_0, \tag{2.2}$$

c belongs to each F_n. □

In the proof, we still get (2.2), provided each F_n is bounded, so that we can define

$$a_n = \inf F_n, \qquad b_n = \sup F_n.$$

If we also have $a_n \in F_n$, then because the F_n compose a decreasing chain, we can conclude

$$\{a_n\colon n \geqslant k\} \subseteq F_k.$$

What property do the F_k need so that, from (2.1), we can conclude $c \in F_k$? It will be the property of being *closed.*

Suppose $A \subseteq \mathbb{R}$ and $b \in A$. We may denote the **complement,** $\mathbb{R} \smallsetminus A$, of A in $\mathbb{R}$ by

$$A^{\mathrm{c}}.$$

There are two possibilities.

1. If, for some positive ε,

$$(b - \varepsilon, b + \varepsilon) \subseteq A,$$

   then b is called an **interior point** of A.
2. In the other case, for all positive ε,

$$(b - \varepsilon, b + \varepsilon) \cap A^{\mathrm{c}} \neq \varnothing,$$

   and now b is a **limit point** of A^{c}. In this case, b is also a limit point of $A^{\mathrm{c}} \cup \{b\}$.

As a special case, if the sequence $(a_n\colon n \in \omega)$ is not eventually constant, and (2.1) holds, then c is a limit point of the set $\{a_n\colon n \in \omega\}$ and therefore of any set that includes this one.

A subset of $\mathbb{R}$ is called

(1) **open,** if its every point is an interior point;

(2) **closed,** if it contains all of its limit points.

Now Lemma 1 generalizes, as discussed.

6 Cantor Intersection Theorem. *Every decreasing chain of nonempty, closed, bounded subsets of $\mathbb{R}$ has nonempty intersection.*

We aim to characterize open and closed sets. Generalizing notation already used, for any *family* $\mathscr{A}$ of sets, we define

$$\bigcup \mathscr{A} = \{x \colon \exists Y\ (Y \in \mathscr{A}\ \ \&\ \ x \in Y)\},$$
$$\bigcap \mathscr{A} = \{x \colon \forall Y\ (Y \in \mathscr{A} \implies x \in Y)\};$$

these are the **union** and **intersection** of $\mathscr{A}$, respectively. A **family** is just a set whose elements are sets whose own elements will be of interest to us.

7 Theorem. *A subset of $\mathbb{R}$ is open if and only if it is the union of a family of open intervals.*

Proof. Open intervals contain all of their interior points, and therefore a union of open intervals does the same.

Conversely, for every open set A, for every b in A, writing

$$E_{A,b} = \{\varepsilon \colon \varepsilon > 0\ \ \&\ \ (b-\varepsilon, b+\varepsilon) \subseteq A\},$$

we have

$$A = \bigcup \bigl\{(b-\varepsilon, b+\varepsilon) \colon b \in A\ \ \&\ \ \varepsilon \in E_{A,b}\bigr\}, \tag{2.3}$$

a union of open intervals. □

The set $\mathbb{Q}$ of rational numbers, and therefore the product $\mathbb{Q} \times \mathbb{Q}$, are countable, and $\mathbb{Q}$ is **dense** in $\mathbb{R}$, in the sense that every interval contains an element of $\mathbb{Q}$.

2 Exercise. Show that the family of open intervals in Theorem 7 can be required to be countable, because, still assuming A is open, we can rewrite (2.3) as

$$A = \bigcup \bigl\{(b-\varepsilon, b+\varepsilon) \colon b \in A \cap \mathbb{Q}\ \ \&\ \ \varepsilon \in E_{A,b} \cap \mathbb{Q}\bigr\}.$$

8 Theorem. *In* $\mathbb{R}$,

(1) if A and B are closed, so is $A \cup B$;

(2) if $\mathscr{F}$ is a family of closed sets, then $\bigcap \mathscr{F}$ is closed;

(3) every closed set is the intersection of a family, each of whose elements is the union of finitely many closed intervals.

3 Exercise. Prove the theorem.

In the theorem, the intersection of the empty family should be understood as $\mathbb{R}$; the union of the empty family is the empty set. Thus $\mathbb{R}$ and $\varnothing$ are closed. Each of them being the complement of the other, they are also open.

According to Theorem 8, the closed intervals of $\mathbb{R}$ compose a **sub-basis,** and their finite unions compose a **basis,** for a **topology** of closed sets on $\mathbb{R}$. By Theorem 7, the open intervals compose a basis of open sets for the same topology. For a precise definition, some writers define a topology on a set to be the family of subsets that are to be called open; other writers, the family of subsets called closed. In any case, a set with a topology is called a **topological space,** and it has open and closed subsets meeting the conditions discussed:

- finite unions of closed sets are closed,
- arbitrary intersections of closed sets are closed,

and complementarily,

- finite intersections of open sets are open,
- arbitrary unions of open sets are open,

2.2 Clopen sets

In any topological space, being the union of the empty set of open sets, the empty set is open; being the union of the empty set of closed sets, it is closed. Then the whole space is also open and closed. We call such sets **clopen.**

9 Theorem. *The only clopen subsets of* $\mathbb{R}$ *are* $\mathbb{R}$ *and* $\varnothing$*.*

Proof. Suppose $A \subseteq \mathbb{R}$ and $b \in A$, but $c \in A^{\mathrm{c}}$. We may assume $b < c$. If b is a limit point of A^{c}, then A is not open. If b is an interior point of A, let

$$d = \sup\{\varepsilon \colon \varepsilon > 0 \ \ \& \ \ [b, b+\varepsilon) \subseteq A\}.$$

Then d is a limit point of both A and A^{c}, so one of these is not closed. □

The intersection of an open subset of $\mathbb{R}$ with C is called open in C; it might not be open in $\mathbb{R}$ (in fact it will not be, unless it is empty). The intersection of a closed subset of $\mathbb{R}$ with C is called closed in C, but it is still closed in $\mathbb{R}$ anyway. In this way we obtain a topology on C, namely the **subspace topology** inherited from $\mathbb{R}$.

Let us recall that C is the intersection of the sets C_n as in (1.5), where

$$\mathrm{C}_n = \bigcup_{\boldsymbol{e}\in {}^{n}2} I_{\boldsymbol{e}}, \tag{2.4}$$

where

$$I_{\boldsymbol{e}} = \left[a_{\boldsymbol{e}}, a_{\boldsymbol{e}} + \frac{1}{3^n}\right], \tag{2.5}$$

where we have

$$a_{\boldsymbol{e}} = \sum_{k\in n} \frac{2e_k}{3^{k+1}}. \tag{2.6}$$

Let us write

$${}^{\omega>}2 = \bigcup_{n\in\omega} {}^{n}2. \tag{2.7}$$

If $\boldsymbol{e} \in {}^{\omega>}2$, let us define

$$\mathrm{D}_{\boldsymbol{e}} = I_{\boldsymbol{e}} \cap \mathrm{C}. \tag{2.8}$$

Each of these is immediately closed in C.

10 Theorem. *The subsets* $\mathrm{D}_{\boldsymbol{e}}$ *of* C *are open and compose a basis of open sets for the subspace topology on* C *inherited from* $\mathbb{R}$.

Proof. For each n in ω,

$$\mathrm{C} = \bigcup_{\boldsymbol{e} \in {}^{n}2} \mathrm{D}_{\boldsymbol{e}}.$$

Since ${}^{n}2$ is finite, and the sets $\mathrm{D}_{\boldsymbol{e}}$ are disjoint from one another, the complement of each of these is closed, so each is open as well as closed.

Now let U be an open subset of $\mathbb{R}$. For every a in $\mathrm{C} \cap U$, for some positive ε,

$$(a - \varepsilon, a + \varepsilon) \subseteq U.$$

With g as in (1.9), we know by Theorem 1 that a is $\mathrm{g}(A)$ for some subset A of ω. Thus

$$a = \sum_{k \in A} \frac{2}{3^{k+1}}.$$

For any n in ω, by (1.10),

$$\begin{aligned} \mathrm{g}(A \cap n) \leqslant a \leqslant \mathrm{g}(A \cap n) + \mathrm{g}(\{x \in \omega \colon x \geqslant n\}) \\ = \mathrm{g}(A \cap n) + \frac{1}{3^n}. \end{aligned}$$

Now let $\boldsymbol{e}$ in ${}^n 2$ be such that

$$\mathrm{g}(A \cap n) = a_{\boldsymbol{e}}$$

as in (2.6). With $I_{\boldsymbol{e}}$ as in (2.5), and now requiring n to be so large that $3^n > 1/\varepsilon$, we have

$$I_{\boldsymbol{e}} \subseteq (a - \varepsilon, a + \varepsilon),$$

and so

$$\mathrm{D}_{\boldsymbol{e}} \subseteq U \cap \mathrm{C}.$$

Since also $a \in \mathrm{D}_{\boldsymbol{e}}$, and a was an arbitrary element of $U \cap \mathrm{C}$, this is a union of various sets $\mathrm{D}_{\boldsymbol{e}}$. □

We now know that unions of finite subsets of

$$\{\mathrm{D}_{\boldsymbol{e}} \colon \boldsymbol{e} \in {}^{\omega >} 2\}$$

are clopen subsets of C. We aim to show the converse. If E is a clopen subset of C, so that, being open, E is a union of *some* family $\mathscr{U}$ of sets $\mathrm{D}_{\boldsymbol{e}}$, we shall show that, being closed, E is $\bigcup \mathscr{U}_0$ for some *finite* subset $\mathscr{U}_0$ of $\mathscr{U}$.

2.3 Compactness

If a topological space is the union of a certain family of open subsets, this family is called an **open cover** of the space. If some finite subset of the family also covers the space, that subset is called a **finite sub-cover.** If every open cover of a space has a finite sub-cover, then the space itself is called **compact**.

Easily, every closed subset of a compact space is compact in the subspace topology, since an open cover of the subset, together with the complement of the subset, yields an open cover of the whole space.

There is an equivalent definition of compactness in terms of closed sets. A family of subsets of a topological space whose every finite subset has nonempty intersection is said to have the **Finite Intersection Property** or **FIP.** Then the space is compact if and only if every family of subsets with FIP has nonempty intersection.

11 Heine–Borel Theorem. *Every closed bounded subset of* $\mathbb{R}$ *is compact in the subspace topology.*

Proof. For reasons discussed, it is enough to show that an interval $[a, b]$ is compact. Writing this interval as I_0, we let $\mathscr{F}$ be a collection of closed subsets of I_0 that has FIP. Let $c = \frac{1}{2}(a + b)$, so that

$$a < c < b.$$

One of the intervals $[a, c]$ and $[c, b]$ is an interval I_1 such that the family $\mathscr{F} \cup \{I_1\}$ has FIP. For, suppose I_1 cannot be $[a, c]$. Then for some finite subset $\mathscr{F}_0$ of $\mathscr{F}$,

$$\bigcap \mathscr{F}_0 \cap [a, c] = \varnothing.$$

But then for every finite subset $\mathscr{F}_1$ of $\mathscr{F}$, since

$$\bigcap(\mathscr{F}_0 \cup \mathscr{F}_1) = \bigcap \mathscr{F}_0 \cap \bigcap \mathscr{F}_1 \neq \varnothing$$

(note that the signs are correct, and we are not applying a De Morgan law), we must have

$$\bigcap \mathscr{F}_0 \cap \bigcap \mathscr{F}_1 \cap [c, b] \neq \varnothing.$$

Thus $\mathscr{F} \cup \{[c, b]\}$ must have FIP.

Continuing in this way, we obtain a decreasing chain of closed intervals I_n such that

$$\mu(I_n) = \frac{1}{2^n}\mu(I_0), \tag{2.9}$$

and each family $\mathscr{F} \cup \{I_n\}$ has FIP. By the Cantor Intersection Theorem, or just Lemma 1, $\bigcap_{n\in\omega} I_n$ contains a point d. By (2.9) then,

$$\bigcap_{n\in\omega} I_n = \{d\}.$$

Then d must be in $\bigcap\mathscr{F}$. For, if it is not, then some E in $\mathscr{F}$ does not contain d. Since E is closed, d is not a limit point of it, and so for some positive ε,

$$E \cap (d-\varepsilon, d+\varepsilon) = \varnothing.$$

If n is so large that $\mu(I_n) < \varepsilon$, then

$$\bigcap\{E, I_n\} = \varnothing,$$

contradicting that $\mathscr{F} \cup \{I_n\}$ has FIP. $\square$

For reasons discussed, we now know that the clopen subsets of C are just the finite unions of sets $\mathrm{D}_{\boldsymbol{e}}$ as in (2.8).

2.4 Continuity

One way that notion of compactness arises in mathematics is as follows. On a subset A of $\mathbb{R}$, a function f is **continuous** if

$$\forall\varepsilon \Bigl(\varepsilon > 0 \implies \forall x \Bigl(x \in A \implies \exists\delta \bigl(\delta > 0 \;\&\; \forall y$$
$$(y \in A \;\&\; |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon)\bigr)\Bigr)\Bigr),$$

while f is **absolutely continuous** if

$$\forall\varepsilon\left(\varepsilon > 0 \implies \exists\delta\left(\delta > 0 \ \& \ \forall x\,(x \in A \implies \forall y\right.\right.$$
$$\left.\left.(y \in A \ \& \ |x-y| < \delta \implies |f(x)-f(y)| < \varepsilon))\right)\right).$$

Note that I write

$$\forall\varepsilon\ (\varepsilon > 0 \implies \dots \text{ rather than } \forall(\varepsilon > 0)\ \dots,$$
$$\exists\delta\ (\delta > 0 \ \& \ \dots \text{ rather than } \exists(\delta > 0)\ \dots$$

4 **Exercise.** Using the Heine–Borel Theorem, prove that, if A is closed and bounded, then every function that is continuous on A is uniformly continuous on A.

It will be useful to note that continuity of f on A is equivalent to the requirement that, for every open set U, $f^{-1}(U)$ be open, and similarly for closed sets.

2.5 König's Lemma

If $\boldsymbol{e} \in {}^n2$, so that the domain of $\boldsymbol{e}$ is n, we may write

$$\operatorname{dom}(\boldsymbol{e}) = n.$$

If $m \leqslant n$, then

$$\boldsymbol{e} \restriction m = (e_0, \dots, e_{m-1}), \tag{2.10}$$

and we may write

$$\boldsymbol{e} \restriction m \subseteq \boldsymbol{e}.$$

When understood to be ordered by inclusion in this way, the set ${}^{\omega>}2$ defined by (2.7) is a **binary tree** of height ω. Note that ${}^{0}2$ has the single element $\varnothing$. For elements of the binary tree, we may use notation as in (2.10), so that if $\sigma \in {}^{\omega}2$ and $n \in \boldsymbol{\omega}$, we write

$$\sigma \restriction n = (\sigma(0), \dots, \sigma(n-1)).$$

If $\boldsymbol{e} \in {}^{n}2$ and $i \in 2$, we may write

$$(\boldsymbol{e}, i) = (e_0, \dots, e_{n-1}, i),$$

an element of ${}^{n+1}2$. In our proof of the Heine–Borel Theorem, we work with the binary tree

$$(I_{\boldsymbol{e}} \colon \boldsymbol{e} \in {}^{\omega>}2),$$

where $I_{\varnothing}$ is the closed interval $[a, b]$, and each $I_{\boldsymbol{e}}$ is divided at its midpoint into the closed intervals $I_{(\boldsymbol{e},0)}$ and $I_{(\boldsymbol{e},1)}$. To be precise, if again $\boldsymbol{e} \in {}^{n}2$, then

$$I_{\boldsymbol{e}} = \left[c_{\boldsymbol{e}}, c_{\boldsymbol{e}} + \frac{b-a}{2^n}\right],$$

where

$$c_{\boldsymbol{e}} = a + (b-a) \cdot \sum_{k \in n} \frac{e_k}{2^{k+1}},$$

so that

$$I_{\boldsymbol{e}} = I_{(\boldsymbol{e},0)} \cup I_{(\boldsymbol{e},1)}, \qquad c_{(\boldsymbol{e},0)} + \frac{b-a}{2^{n+1}} = c_{(\boldsymbol{e},1)}.$$

We find σ in ${}^{\omega>}2$ such that, for each n in $\boldsymbol{\omega}$, the family

$$\mathscr{F} \cup \{I_{\sigma \restriction n}\}$$

has FIP. We can do this because, if $\mathscr{F} \cup \{I_{\boldsymbol{e}}\}$ has FIP, then so does one of $\mathscr{F} \cup \{I_{(\boldsymbol{e},0)}\}$ and $\mathscr{F} \cup \{I_{(\boldsymbol{e},1)}\}$. The same idea yields the following.

12 König's Lemma. *If T is an infinite subset of ${}^{\omega}2$, then for some σ in ${}^{\omega}2$, the set*

$$\{n \in \omega \colon \sigma \restriction n \in T\}$$

is infinite. More generally, every infinite finitely branching tree *has an infinite branch.*

Proof. If, for some n in ω, for some $\boldsymbol{e}$ in ${}^{n}2$, the set

$$\{\boldsymbol{x} \in T \colon \boldsymbol{e} \subseteq \boldsymbol{x}\}$$

is infinite, then, for some i in 2, the set

$$\{\boldsymbol{x} \in T \colon (\boldsymbol{e}, i) \subseteq \boldsymbol{x}\}$$

must be infinite, since

$$\{\boldsymbol{x} \in T \colon \boldsymbol{e} \subseteq \boldsymbol{x}\} \subseteq \{\boldsymbol{e}\} \cup \bigcup_{i \in 2} \{\boldsymbol{x} \in T \colon (\boldsymbol{e}, i) \subseteq \boldsymbol{x}\}.$$

If perhaps not 2, but a finite number of elements of the tree are immediately above a given element, the same argument works. □

In the same way, we can prove directly that C is compact. For suppose $\mathscr{F}$ is a family of subsets of C having FIP. In the notation of (2.8), we have

$$\mathrm{C} = \mathrm{D}_{\varnothing},$$

and for all n in ω, for all $\boldsymbol{e}$ in ${}^{n}2$,

$$\mathrm{D}_{\boldsymbol{e}} = \mathrm{D}_{(\boldsymbol{e},0)} \cup \mathrm{D}_{(\boldsymbol{e},1)}.$$

Then trivially $\mathscr{F} \cup \{\mathrm{D}_\varnothing\}$ has FIP, and if $\mathscr{F} \cup \{\mathrm{D}_{\boldsymbol{e}}\}$ has FIP, then so must $\mathscr{F} \cup \{\mathrm{D}_{(\boldsymbol{e},i)}\}$ for some i in 2. Hence for some σ in ${}^{\omega}2$, for each n in ω, each set $\mathscr{F} \cup \{\mathrm{D}_{\sigma\restriction n}\}$ has FIP. Since, with g as in (1.9), with σ^{-1} as in (1.16) we have

$$\bigcap_{n\in\omega} \mathrm{D}_{\sigma\restriction n} = \left\{ \sum_{\sigma(k)=1} \frac{2}{3^{k+1}} \right\} = \left\{ \mathrm{g}(\sigma^{-1}(1)) \right\},$$

we can conclude as before

$$\mathrm{g}(\sigma^{-1}(1)) \in \bigcap \mathscr{F}.$$

2.6 Tychonoff topology

As in (1.9) we defined the map g from $\mathscr{P}(\omega)$ to $\mathbb{R}$ that turned out, by Theorems 1 and 2, to be a bijection onto C, so now we let G be the bijection from ${}^{\omega}2$ to C given by

$$\mathrm{G}(\sigma) = \sum_{k\in\omega} \frac{2\sigma(k)}{3^{k+1}}.$$

For each n in ω, for each i in 2, let us define

$$E_{n\,i} = \{\sigma \in {}^{\omega}2 \colon \sigma(n) = i\},$$

so that

$$\mathrm{G}[E_{n\,i}] = \bigcup_{\boldsymbol{e}\in{}^{n}2} \mathrm{D}_{(\boldsymbol{e},i)},$$

and also, for each $\boldsymbol{e}$ in ${}^{n}2$,

$$\mathrm{G}^{-1}(\mathrm{D}_{\boldsymbol{e}}) = \{\sigma \in {}^{\omega}2 \colon \sigma \restriction n = \boldsymbol{e}\} = \bigcap_{k\in n} E_{k\,e_k}.$$

Therefore the sets $E_{n\,i}$ compose a sub-basis of closed sets for a topology on ${}^{\omega}2$ in which G is a **homeomorphism** onto C. This means G is a bijection, and both G and G^{-1} are *continuous.* Generalizing from the description of continuity on page 34, we define a function f from one topological space to another to be **continuous** if $f^{-1}(F)$ is closed for every closed set F.

We give the set 2 the topology in which every subset is closed. Then the topology on ${}^{\omega}2$ is the **product topology** or **Tychonoff topology** for the following reason.

If $(\Omega_i : i \in I)$ is an indexed family of topological spaces, we define the product

$$\prod_{i \in I} \Omega_i$$

as the set of functions f from I to $\bigcup_{i \in I} \Omega_i$ such that, for each i in I,

$$f(i) \in \Omega_i.$$

If we denote the product by Ω, we can define, for each i in I, the function π_i from Ω to Ω_i given by

$$\pi_i(f) = f(i).$$

Here Ω is the capital Greek letter Omega.* The Tychonoff topology on Ω is the weakest topology in which each function π_i is continuous. This means that the sets ${\pi_i}^{-1}(F)$, where F ranges over the closed subsets of Ω_i, where i ranges over I, compose a sub-basis of closed subsets of Ω. In the special case when I is ω, and each Ω_i is 2, we recover the topology on ${}^{\omega}2$ that we have already defined.

*Often the Capital letter X is used for a topological space, but I prefer to use that letter as a variable for sets.

13 Tychonoff Theorem. *Every product of compact spaces is compact in the Tychonoff topology.*

Proof. In the notation above, let $\mathscr{F}$ be family of closed subsets of Ω that has FIP. Each element F of $\mathscr{F}$ is the intersection of a family $\mathscr{E}_F$ of sub-basic closed subsets. Now let

$$\mathscr{F}^* = \bigcup_{F\in\mathscr{F}} \mathscr{E}_F.$$

This too has FIP, and

$$\bigcap \mathscr{F}^* = \bigcap \mathscr{F}.$$

Suppose now $\mathscr{F}^*$ contains

$$\bigcup_{k\in n} {\pi_{i(k)}}^{-1}(F_k)$$

for some $(i(k)\colon k \in n)$ in ${}^nI$ and some closed subset F_k of $\Omega_{i(k)}$ for each k in n. As before, for some k in n, the set

$$\mathscr{F}^* \cup \{{\pi_{i(k)}}^{-1}(F_k)\}$$

must have FIP; moreover, the intersection of this set is included in $\bigcap \mathscr{F}^*$ itself. By continuing in this way—strictly we use **Zorn's Lemma,** which follows from the Axiom of Choice)—, we obtain a set $\mathscr{G}$ with FIP such that

$$\bigcap \mathscr{G} \subseteq \bigcap \mathscr{F}^*,$$

and whose every element is, for some i in I, of the form ${\pi_i}^{-1}(F)$, where F is a closed subset of Ω_i. For each i in I, let $\mathscr{G}_i$ consist of the closed subsets F of Ω_i such that ${\pi_i}^{-1}(F)$ belongs to $\mathscr{G}$. Then $\mathscr{G}_i$ has FIP, so $\bigcap \mathscr{G}_i$ has an element a_i, if we assume Ω_i is compact. Then

$$(a_i\colon i \in I) \in \bigcap \mathscr{G}. \qquad \square$$

3 Algebra

3.1 Logic

The Tychonoff topology on ${}^{\omega}2$ arises independently in propositional logic as follows.

Starting with a collection $\{P_k : k \in \omega\}$ of **(propositional) variables,** we define **(propositional) formulas** recursively:

1. Each variable is a formula, namely an **atomic** formula.
2. If F is a formula, then so is $\neg G$, the **negation** of F.
3. If F and G are formulas, then so is $(F \wedge G)$, the **conjunction** of F and G.

Let us denote by

$$\mathrm{L}$$

the set of propositional formulas so defined. Then for example, if L contains F and G, then it contains

$$\neg(\neg F \wedge \neg G),$$

which is the **disjunction** of F and G; we may denote this by $(F \vee G)$. Then the expression

$$(\neg F \vee G)$$

stands for another formula in L, denoted by $F \to G$. Officially though, L is the smallest set that includes $\{P_k : k \in \omega\}$ and is closed under the operations $F \mapsto \neg F$ and $(F, G) \mapsto (F \wedge G)$.

2 Lemma. *Every formula in* L *is* ***uniquely readable:***

1) *no atomic formula is also a negation or a conjunction;*
2) *no negation is also a conjunction;*
3) *every conjunction is uniquely so.*

Proof. Only the last claim is not entirely clear. By induction in L, we show that for all formulas F,

(a) no proper initial segment of F is a formula, and
(b) F is not a proper initial segment of any formula.

1. The claim is clearly true when F is atomic.

2. Suppose the claim is true when F is a formula G. Then the claim must be true when F is $\neg G$. For if H is a proper initial segment of $\neg G$, then H is of the form $\neg K$ for some K, which is a proper initial segment of G, so, by hypothesis, K cannot be a formula, and therefore H cannot be a formula. There is a similar argument if $\neg G$ is a proper initial segment of H.

3. Similarly, if the claim is true when F is G or H, then it must be true when F is $(G \wedge H)$.

By induction, which is made possible by the recursive definition of L, the claim holds for all formulas F. The last part of the theorem now follows. □

Lemma 2 allows us to make recursive definitions of functions on L. For example, for all subsets A of ω, we recursively define which formulas are **true** in A. We shall express that a formula F is true in A by writing

$$A \vDash F.$$

Then by definition

$$A \vDash P_k \iff k \in A, \tag{3.1a}$$
$$A \vDash \neg F \iff A \nvDash F, \tag{3.1b}$$
$$A \vDash (F \wedge G) \iff A \vDash F \ \ \& \ \ A \vDash G. \tag{3.1c}$$

Note that the expressions $\iff$ and $\&$ here, as throughout these notes, are just abbreviations of ordinary language.

Without recursion, if $F \in \mathrm{L}$, we define

$$\mathrm{Mod}(F) = \{X \in \mathscr{P}(\omega)\colon X \vDash F\}. \tag{3.2}$$

If $\Gamma \subseteq \mathrm{L}$, we define

$$\mathrm{Mod}(\Gamma) = \bigcap\{\mathrm{Mod}(F)\colon F \in \Gamma\}; \tag{3.3}$$

this is the set of **models** of Γ.

3 Lemma. *The family*

$$\{\mathrm{Mod}(\Gamma)\colon \Gamma \subseteq \mathrm{L}\}$$

is the family of closed subsets of $\mathscr{P}(\omega)$ in the topology whereby the function $X \mapsto \chi_X$, which is given by (1.15) and whose inverse $\sigma \mapsto \sigma^{-1}(1)$ is given by (1.16), is a homeomorphism with ${}^{\omega}2$ in the Tychonoff topology. The clopen subsets of the topology on $\mathscr{P}(\omega)$ compose the family

$$\{\mathrm{Mod}(F)\colon \Gamma \in \mathrm{L}\}.$$

Proof. The family of sets $\mathrm{Mod}(F)$ is a basis of closed sets for a topology, since the family is closed under taking finite unions. Indeed the family contains the empty set as

$$\mathrm{Mod}(F \wedge \neg F),$$

and the family is closed under binary unions since

$$\operatorname{Mod}(F) \cup \operatorname{Mod}(G) = \operatorname{Mod}(F \vee G). \tag{3.4}$$

Moreover,

$$\begin{aligned} &\{X \in \mathscr{P}(\omega) \colon \chi_X \in E_{k\,1}\} \\ &\qquad\qquad = \{X \in \mathscr{P}(\omega) \colon k \in X\} = \operatorname{Mod}(P_k), \end{aligned}$$

and likewise

$$\{X \in \mathscr{P}(\omega) \colon \chi_X \in E_{k\,0}\} = \operatorname{Mod}(\neg P_k),$$

so $X \mapsto \chi_X$ is continuous. Since in addition to (3.4) we have generally

$$\operatorname{Mod}(F) \cap \operatorname{Mod}(G) = \operatorname{Mod}(F \wedge G), \tag{3.5}$$

$$\operatorname{Mod}(F)^{\mathrm{c}} = \operatorname{Mod}(\neg F), \tag{3.6}$$

by means of the De Morgan laws we obtain that each set $\operatorname{Mod}(F)$ is a finite union of finite intersections of sets $\operatorname{Mod}(P_k)$ and $\operatorname{Mod}(\neg P_k)$. This yields that $\sigma \mapsto \sigma^{-1}(1)$ is continuous, and the rest follows. □

If every finite subset of Γ has a model, we shall say that Γ is **consistent.** This is another way of saying that the family $\{\operatorname{Mod}(F) \colon F \in \Gamma\}$ has FIP. We shall show that every consistent set of formulas has a model. This is another way of saying $\mathscr{P}(\omega)$ is compact. We already know this compactness, but now we shall prove it in a new way, or at least in new language.

To do so, we make one more recursive definition, parallel to (3.1).

$$V(P_k) = \{k\}, \tag{3.7a}$$
$$V(\neg F) = V(F), \tag{3.7b}$$
$$V\big((F \wedge G)\big) = V(F) \cup V(G). \tag{3.7c}$$

We do not really need recursion here: we can just say

$$V(F) = \{k \in \omega \colon P_k \text{ occurs in } F\}.$$

We are defining a formal logic. Logic lets us do mathematics with logical precision. Such precision may be illusory when used to define the logic in the first place. The same consideration arises in the proof of the following.

4 **Lemma.** *Let F be a formula, and let A and B be subsets of ω such that*

$$V(F) \cap A = V(F) \cap B.$$

Then

$$A \vDash F \iff B \vDash F.$$

Proof. The theorem is that whether F is true in A depends only on whether $k \in A$ when P_k actually occurs in F. This is obvious when F is an atomic formula, by (3.1a). The remaining rules (3.1b) and (3.1c) maintain the claim, since they do not involve variables explicitly. In all formal detail though, we can use induction to show

$$A \vDash F \iff V(F) \cap A \vDash F$$

as follows.

1. Supposing first that F is an atomic formula P_k, we have $V(F) = \{k\}$ by (3.7a), and then

$$\begin{aligned} A \vDash F &\iff k \in A && \text{[by (3.1a)]} \\ &\iff k \in V(F) \cap A && \text{[by (3.7a)]} \\ &\iff V(F) \cap A \vDash F. && \text{[by (3.1a) again]} \end{aligned}$$

2. Suppose the claim is true when F is a formula G. Then

$$\begin{aligned} A \vDash \neg G &\iff A \nvDash G && \text{[by (3.1b)]} \\ &\iff V(G) \cap A \nvDash G && \text{[by hypothesis]} \\ &\iff V(\neg G) \cap A \nvDash G && \text{[by (3.7b)]} \\ &\iff V(\neg G) \cap A \vDash \neg G, && \text{[by (3.1b) again]} \end{aligned}$$

so the claim holds when F is $\neg G$.

3. Suppose finally the claim is true when F is either of G and H. Since

$$V(G) \subseteq V\big((G \wedge H)\big), \qquad V(H) \subseteq V\big((G \wedge H)\big) \tag{3.8}$$

by (3.7c), so that

$$\begin{aligned} V(G) \cap V\big((G \wedge H)\big) \cap A = V(G) \cap A, \\ V(H) \cap V\big((G \wedge H)\big) \cap A = V(H) \cap A, \end{aligned} \tag{3.9}$$

we have

$$\begin{aligned} A \vDash G &\iff V\big(G \wedge H)\big) \cap A \vDash G, \\ A \vDash H &\iff V\big(G \wedge H)\big) \cap A \vDash H, \end{aligned} \tag{3.10}$$

since for example

$$\begin{aligned} A \vDash G &\iff V(G) \cap A \vDash G && \text{[by hyp.]} \\ &\iff V(G) \cap V\big((G \wedge H)\big) \cap A \vDash G && \text{[by (3.9)]} \\ &\iff V\big((G \wedge H)\big) \cap A \vDash G. && \text{[by hyp.]} \end{aligned}$$

This gives us

$$
\begin{aligned}
A \vDash (G \wedge H) &\iff A \vDash G \ \ \& \ \ A \vDash H && [(3.1c)]\\
&\iff V\big((G \wedge H)\big) \cap A \vDash G\\
&\qquad\qquad \& \ \ V\big((G \wedge H)\big) \cap A \vDash H && [(3.10)]\\
&\iff V\big((G \wedge H)\big) \cap A \vDash (G \wedge H). && [(3.1c)]
\end{aligned}
$$

This completes the induction. □

14 Compactness Theorem. *Every consistent set of propositional formulas has a model.*

Proof. Let Γ be a consistent set of formulas. As in our earlier proofs, one of $\Gamma \cup \{P_0\}$ and $\Gamma \cup \{\neg P_0\}$ must be consistent. In this way, by recursion, we obtain a sequence $(G_k : k \in \omega)$, where each G_k is either P_k or $\neg P_k$, and each collection $\Gamma \cup \{G_k : k < n\}$ is consistent. Let

$$A = \{k \in \omega : G_k \text{ is } P_k\}.$$

For all F in Γ, the collection

$$\{F\} \cup \{G_k : k \in V(F)\},$$

being finite, has a model B. Then for all k in $V(F)$, we have $B \vDash G_k$, and so

$$k \in B \iff k \in A.$$

By the last lemma, since $B \vDash F$, also $A \vDash F$. Thus $A \in \mathrm{Mod}(\Gamma)$. □

3.2 Boolean algebras

The relation $\sim$ of **logical equivalence** on L is given by

$$F \sim G \iff \mathrm{Mod}(F) = \mathrm{Mod}(G). \tag{3.11}$$

We can now define

$$[F] = \{X \in \mathrm{L} \colon X \sim F\},$$
$$\mathrm{L}/{\sim} = \{[X] \colon X \in \mathrm{L}\}.$$

The definitions ensure that there is a well-defined injection

$$[X] \mapsto \mathrm{Mod}(X)$$

from $\mathrm{L}/{\sim}$ to $\mathscr{P}(\omega)$. By Lemma 3, the map is also surjective onto the collection $\mathscr{B}$ of clopen subsets of ω. Here $\mathscr{B}$ is closed under

(1) the binary operations $\cap$ and $\cup$,
(2) the singulary operation ${}^{\mathrm{c}}$, and
(3) the nullary operations $\varnothing$ and ω (that is, $\mathscr{B}$ contains these sets).

This makes $\mathscr{B}$ a *Boolean subalgebra* of $\mathscr{P}(\omega)$. Because of (3.4), (3.5), and (3.6), along with

$$\mathrm{Mod}(P_0 \wedge \neg P_0) = \varnothing,$$
$$\mathrm{Mod}(P_0 \vee \neg P_0) = \omega,$$

$\mathrm{L}/{\sim}$ is a Boolean algebra with respect to the (well-defined) operations given by

$$[F] \wedge [G] = [F \wedge G],$$
$$[F] \vee [G] = [F \vee G],$$
$$\neg[F] = [\neg F],$$
$$\bot = [P_0 \wedge \neg P_0],$$
$$\top = [P_0 \vee \neg P_0].$$

In general, an abstract **Boolean algebra** (like L/$\sim$) is a set B with operations $\wedge$, $\vee$, $\neg$, $\perp$, and $\top$ with the following properties.

1. The binary operations $\vee$ and $\wedge$ are **commutative:**

$$x \vee y = y \vee x, \qquad x \wedge y = y \wedge x.$$

2. The elements $\perp$ and $\top$ are **identities** for $\vee$ and $\wedge$ respectively:

$$x \vee \perp = x, \qquad x \wedge \top = x.$$

3. $\vee$ and $\wedge$ are mutually **distributive:**

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z), \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

4. The element $\neg x$ is a **complement** of x:

$$x \vee \neg x = \top, \qquad x \wedge \neg x = \perp.$$

Additional properties like associativity of $\wedge$ and $\vee$ follow from the given identities:

5 Lemma (E. Huntington, 1904)**.** *In any Boolean algebra:*

$$\begin{aligned} x \vee x &= x, & x \wedge x &= x, \\ x \vee \top &= \top, & x \wedge \perp &= \perp, \\ x \vee (x \wedge y) &= x, & x \wedge (x \vee y) &= x, \\ \neg\neg x &= x, & & \\ \neg(x \vee y) &= \neg x \wedge \neg y, & \neg(x \wedge y) &= \neg x \vee \neg y, \\ (x \vee y) \vee z &= x \vee (y \vee z), & (x \wedge y) \wedge z &= x \wedge (y \wedge z). \end{aligned} \tag{3.12}$$

Proof. We prove only (3.12):

$$\begin{aligned} x \vee (x \wedge y) &= (x \wedge \top) \vee (x \wedge y) \\ &= x \wedge (\top \vee y) = x \wedge \top = x. \end{aligned}$$ □

We shall investigate how $\mathscr{P}(\omega)$, or more precisely a space homeomorphic with it, can be obtained from the Boolean algebra $\mathrm{L}/{\sim}$. The same construction will work for any Boolean algebra, and then the algebra can be recovered as being isomorphic to the algebra of clopen subsets of the space. This is the **Stone Representation Theorem,** Theorem 15 below.

We shall need that, on any Boolean algebra, there is a partial ordering $\vdash$ given by

$$x \vdash y \iff x \wedge y = x.$$

Note that, by (3.12),

$$x \wedge y = x \iff x \vee y = y.$$

If $A \subseteq \omega$, we define

$$\mathrm{Th}(A) = \{X \in \mathrm{L} \colon A \vDash X\}. \tag{3.13}$$

This is the **theory** of A, and it has the following properties.

$$F \in \mathrm{Th}(A) \ \& \ G \in \mathrm{Th}(A) \implies (F \wedge G) \in \mathrm{Th}(A), \tag{3.14}$$

$$F \in \mathrm{Th}(A) \ \& \ F \vdash G \implies G \in \mathrm{Th}(A), \tag{3.15}$$

$$F \notin \mathrm{Th}(A) \iff \neg F \in \mathrm{Th}(A). \tag{3.16}$$

As a result, $\mathrm{Th}(A)$, or more precisely the set $\{[X] \colon X \in \mathrm{Th}(A)\}$,

- is a **filter** of the algebra $\mathrm{L}/{\sim}$, by (3.14) and (3.15),

- by these and (3.16), is an **ultrafilter.**

Note that we can replace (3.14) and (3.15) with the single equivalence

$$F \in \mathrm{Th}(A) \;\;\&\;\; G \in \mathrm{Th}(A) \iff (F \wedge G) \in \mathrm{Th}(A).$$

We may blur the distinction between formulas in L and their equivalence classes in L/$\sim$, thus identifying the sets $\mathrm{Th}(A)$ and $\{[X] \colon X \in \mathrm{Th}(A)\}$. An ultrafilter is a maximal filter, if the whole algebra is not counted as a filter; any larger filter than $\mathrm{Th}(A)$ would contain some F not in $\mathrm{Th}(A)$; but then $\mathrm{Th}(A)$ contains $\neg F$, so the larger filter contains $F \wedge \neg F$, which is equivalent to $\bot$, and $\bot \vdash G$ for all G in L.

The converse also holds:

6 Lemma. *Every ultrafilter of* L/$\sim$ *is the theory of some element of* $\mathscr{P}(\omega)$.

Proof. Given an ultrafilter Φ of L/$\sim$, we may let

$$A = \{k \in \omega \colon P_k \in \Phi\}.$$

By induction in L, $\Phi = \mathrm{Th}(A)$, that is, for all F in L,

$$F \in \Phi \iff A \vDash F. \tag{3.17}$$

In detail:

1. (3.17) holds by (3.1a) when F is atomic.
2. If (3.17) holds when F is G, then

$$\begin{aligned} \neg G \in \Phi &\iff G \notin \Phi && \text{[by (3.16)]} \\ &\iff A \nvDash G && \text{[by hypothesis]} \\ &\iff A \vDash \neg G. && \text{[by (3.1b)]} \end{aligned}$$

3. If (3.17) holds when F is either of G and H, then

$$\begin{aligned}
(G \wedge H) \in \Phi &\iff G \in \Phi \ \& \ H \in \Phi && \text{[by (3.14) and (3.15)]}\\
&\iff A \vDash G \ \& \ A \vDash H && \text{[by hypothesis]}\\
&\iff A \vDash (G \wedge H). && \text{[by (3.1c)]}
\end{aligned}$$

This completes the induction. □

3.3 Stone spaces

For any Boolean algebra B, we shall denote the set of ultrafilters of B by

$$\mathrm{S}(B);$$

this is the **Stone space** of B, because it will have a topology. In our case, we have a bijection $X \mapsto \mathrm{Th}(X)$ from $\mathscr{P}(\omega)$ to $\mathrm{S}(\mathrm{L}/{\sim})$, and this will be a homeomorphism.

We have been working with the relation $\vDash$ from $\mathscr{P}(\omega)$ to L. We have used it to define, by (3.2), a map $X \mapsto \mathrm{Mod}(X)$ from L to $\mathscr{P}(\mathscr{P}(\omega))$. This map has the properties given by (3.5) and (3.6), so that, when we define the relation $\sim$ of logical equivalence as in (3.11), the map $X \mapsto \mathrm{Mod}(X)$ induces a Boolean-algebra embedding of $\mathrm{L}/{\sim}$ in $\mathscr{P}(\mathscr{P}(\omega))$. By Lemma 3, the embedding is an isomorphism with the algebra of clopen subsets of $\mathscr{P}(\omega)$.

We have also defined by (3.13) a "dual" map, $Y \mapsto \mathrm{Th}(Y)$, from $\mathscr{P}(\omega)$ to $\mathscr{P}(\mathrm{L}/{\sim})$. By Lemma 6, the map is a bijection onto $\mathrm{S}(\mathrm{L}/{\sim})$, the set of ultrafilters of $\mathrm{L}/{\sim}$.

As by (3.3) we define $\mathrm{Mod}(\Gamma)$ when $\Gamma \subseteq \mathrm{L}$, so we can define

$$\mathrm{Th}(\mathscr{A}) = \bigcap_{Y \in \mathscr{A}} \mathrm{Th}(Y)$$

when $\mathscr{A} \subseteq \mathscr{P}(\omega)$. Then

$$\Gamma \subseteq \Delta \implies \operatorname{Mod}(\Gamma) \supseteq \operatorname{Mod}(\Delta),$$
$$\mathscr{A} \subseteq \mathscr{B} \implies \operatorname{Th}(\mathscr{A}) \supseteq \operatorname{Th}(\mathscr{B}).$$

Simply on this basis,

$$\operatorname{Mod} \circ \operatorname{Th} \circ \operatorname{Mod} = \operatorname{Mod},$$
$$\operatorname{Th} \circ \operatorname{Mod} \circ \operatorname{Th} = \operatorname{Th},$$

so that there is a one-to-one correspondence, called a **Galois correspondence,** between the sets $\operatorname{Mod}(\Gamma)$ and the sets $\operatorname{Th}(\mathscr{A})$. In the original Galois theory, the correspondence is between subfields of a field K and subgroups of the group of automorphisms of K; one obtains this by using, in place of our $\models$, the relation R from the field to the automorphism group given by

$$x \mathrel{R} \sigma \iff x^{\sigma} = x.$$

In our case, the sets $\operatorname{Th}(\mathscr{A})$ are filters of $\mathrm{L}/{\sim}$, while the sets $\operatorname{Mod}(\Gamma)$ can be called **elementary classes** (though all this means is that they are the classes of models of sets of formulas).

15 Stone Representation Theorem. *Every Boolean algebra embeds in an algebra $\mathscr{P}(\Omega)$ for some set Ω. Indeed, when the algebra is B, it embeds in $\mathscr{P}(\mathrm{S}(B))$ under the map $x \mapsto [x]$, where*

$$[a] = \{U \in \mathrm{S}(B) \colon a \in U\};$$

and then the subsets $[a]$ of $\mathrm{S}(B)$ are the clopen sets in a compact topology on $\mathrm{S}(B)$.

Proof. In what we have done, if we replace L with an arbitrary Boolean algebra B, then we can also replace $\mathscr{P}(\omega)$ with $\mathrm{S}(B)$, and $\vDash$ with $\in$. When we define the map $x \mapsto [x]$ from B to $\mathscr{P}(\mathrm{S}(B))$ as indicated, then

$$[a] \cap [b] = [a \wedge b],$$
$$[a]^{\mathrm{c}} = [\neg a],$$

so the map is a homomorphism of Boolean algebras. It is an embedding, by the Axiom of Choice, or rather by the weaker axiom called the **Prime Ideal Theorem** (which is that every proper ideal of a ring is included in a prime ideal; the Axiom of Choice gives that every proper ideal is included in a maximal ideal; maximal ideals are always prime, but in Boolean rings, prime ideals are also maximal).

Since the image of B in $\mathscr{P}(\mathrm{S}(B))$ is closed under finite unions, it is a basis for a topology of closed sets on $\mathrm{S}(B)$. Thus the closed sets in this topology are the sets

$$\bigcap_{x \in I} [x],$$

where $I \subseteq B$. The topology is compact, because if $\bigcap_{x \in I_0} [x]$ is never empty when I_0 is a finite subset of I, then I is included in a proper filter, and therefore (by the Prime Ideal Theorem) an ultrafilter U; but this just means $U \in \bigcap_{x \in I} [x]$. $\square$

The topology in the theorem is the **Stone topology** on $\mathrm{S}(B)$.

A **first-order logic,** such as the logic of set theory, defines formulas as in propositional logic, except that the atomic formulas are not propositional variables, but (in the case of set theory) formulas $x \in y$; also, if φ is a formula, and x is a variable, then so is $\exists x\ \varphi$. One has the notion of a **free variable** of a formula; if a formula has no free variable, the

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Figure 3.1: Ring operations on 2

formula is a **sentence.** Every sentence has a class of **models,** which, considered in themselves, are **structures.** Defining logical equivalence as before, one obtains a Boolean algebra of sentences, called a **Lindenbaum algebra** after a student of Tarski, murdered by the Nazis. The Stone space of this algebra is automatically compact. The Compactness Theorem of first-order logic can then be understood as corresponding to Lemma 6: it is that every ultrafilter of the Lindenbaum algebra is in fact the theory of some structure (and it is not so easy to prove).

Finally, in the Stone Representation Theorem, instead of a Boolean algebra, we may start with an arbitrary topological space and extract its algebra of clopen subsets. However, the Stone space of this algebra will not be homeomorphic with the original space unless this is compact and **totally disconnected** (for any two points, some clopen set contains only one of them).

3.4 Topological rings

With respect to the operations of addition and multiplication defined unsurprisingly as in Figure 3.1, the set 2 becomes a

(commutative) ring because the following identities hold:

$$
\begin{aligned}
x + y &= y + x, & xy &= yx,\\
x + (y + z) &= (x + y) + z, & x(yz) &= (xy)z,\\
x + 0 &= x, & x \cdot 1 &= x,
\end{aligned}
$$

$$x(y + z) = xy + xz,$$

$$x + x = 0.$$

For an arbitrary commutative ring, the last identity would be $x - x = 0$; but $-x = x$ in 2. Also in 2,

$$x^2 = x,$$

and therefore 2 is called a **Boolean ring.** From this and the other identities, $x + x = 0$ and $xy = yx$ follow, at least if additive cancellation is assumed, so that

$$x^2 + y^2 = x + y = (x + y)^2 = x^2 + xy + yx + y^2$$

and therefore

$$0 = xy + yx,$$

and hence as a special case

$$0 = x^2 + x^2 = x + x,$$

and consequently

$$xy = xy + yx + yx = yx.$$

A Boolean ring is a Boolean algebra, and conversely, by the rules

$$x \wedge y = xy,$$

$$x \vee y = x + y + xy,$$

$$\neg x = x + 1,$$

$$(x \vee y) \wedge (\neg x \vee \neg y) = x + y.$$

Now ${}^{\omega}2$ is a Boolean ring with respect to the pointwise operations given by

$$(x_k \colon k \in \omega) * (y_k \colon k \in \omega) = (x_k * y_k \colon k \in \omega),$$

where $*$ is $+$ or $\times$. By the Stone Representation Theorem, ${}^{\omega}2$ is isomorphic to the Boolean algebra of clopen subsets in the Stone topology on $\mathrm{S}({}^{\omega}2)$. This space consists of the **principal ultrafilters**

$$\{\chi_X \colon a \in X\},$$

where $a \in \omega$, along with the **nonprincipal ultrafilters,** which include the filter

$$\{\chi_X \colon X^{\mathrm{c}} \in \mathscr{P}_{\omega}(\omega)\},$$

where $\boxed{\mathscr{P}_{\omega}(A)}$ consists of the finite subsets of A; but the existence of the nonprincipal ultrafilters depends on the Prime Ideal Theorem mentioned above.

The elements $\chi_{\{n\}}$ (where $n \in \omega$) are the **atoms** of the Boolean algebra ${}^{\omega}2$; but when this is given the Tychonoff topology, the Boolean algebra of clopen subsets is atomless.

The Boolean ring ${}^{\omega}2$ is a **topological ring** with respect to the Tychonoff topology, because addition and multiplication are continuous in each factor. Indeed, if $A \in \mathscr{P}(\omega)$, and

- if f is $x \mapsto x + \chi_A$, then

$$f^{-1}(E_{n\,i}) = \begin{cases} E_{n\,i+1}, & \text{if } n \in A, \\ E_{n\,i}, & \text{if } n \in A^{\mathrm{c}}; \end{cases}$$

- if g is $x \mapsto x \cdot \chi_A$, then

$$f^{-1}(E_{n\,i}) = \begin{cases} E_{n\,i}, & \text{if } n \in A, \\ \varnothing, & \text{if } n \in A^{\mathrm{c}} \;\; \& \;\; i = 1; \\ {}^{\omega}2, & \text{if } n \in A^{\mathrm{c}} \;\; \& \;\; i = 0. \end{cases}$$

There is another ring structure on ${}^{\omega}2$ that makes this a topological ring with respect to the Tychonoff topology. In the new ring structure, whether $*$ is $+$ or $\times$, we have

$$(x_k : k \in \omega) * (y_k : k \in \omega) = (z_k : k \in \omega)$$

if and only if, for all n in ω,

$$\sum_{k\in n} x_k 2^k * \sum_{k\in n} y_k 2^k \equiv \sum_{k\in n} z_k 2^k \pmod{2^n},$$

the computations being performed in ω as usual. The element $(x_k : k \in \omega)$ can be denoted by

$$\sum_{k\in\omega} x_k 2^k,$$

and the new ring itself by $\boxed{\mathbb{Z}_{(2)}}$. This is the ring of **dyadic integers** (or 2-adic integers). Since

$$1 + \sum_{k\in\omega} 2^k = 0$$

in the ring, negatives do exist. If a is a nonzero element $\sum_{k\in\omega} a_k 2^2$ of $\mathbb{Z}_{(2)}$, we define

$$|a| = \frac{1}{2^m},$$

where

$$m = \min\{n \in \omega : a_n \neq 0\}.$$

Also $|0| = 0$. Then the **triangle inequality** holds in the strong form

$$|x + y| \leqslant \max\{|x|, |y|\},$$

so the function $(x, y) \mapsto |x - y|$ is a *metric* on $\mathbb{Z}_{(2)}$ (called an *ultrametric*). The induced topology is just the Tychonoff topology, since, if we define

$$\mathrm{B}(a; r) = \{x \in \mathbb{Z}_{(2)} \colon\ |x - a| < r\},$$

so that these are a basis of open sets in $\mathbb{Z}_{(2)}$, then

$$E_{n\ i} = \bigcup_{\boldsymbol{e} \in {}^n 2} \mathrm{B}\left(\sum_{k \in n} e_k 2^k + i2^n; \frac{1}{2^n}\right),$$

$$\mathrm{B}\left(\sum_{k \in \omega} a_k 2^k; \frac{1}{2^n}\right) = \bigcap_{k \in n} E_{k\ a_k}.$$