A course in projective geometry

Projektif geometri dersi

David Pierce
Matematik Bölümü
Mimar Sinan Güzel Sanatlar Üniversitesi
mat.msgsu.edu.tr/~dpierce/
dpierce@msgsu.edu.tr

September 12–18, 2016
Last edited January 9, 2017

Nesin Matematik Köyu
Şirince, Selçuk, İzmir

Preface

This is about a two-week course called *Geometriler* at the Nesin Mathematics Village. In the first week we read, in my Turkish translation, some of the lemmas for Euclid's *Porisms* in Book VII of the *Collection* of Pappus. Students presented the lemmas at the board, and I lectured on modern developments due to Desargues and others. In the following week, we read Lobachevski's "Geometrical Researches on the Theory of Parallels" in Halstead's English translation. Again students presented propositions at the board; but curved space turned out to be a more difficult concept than points at infinity. That week is covered only briefly here, in the Introduction.

I had given a similar *Geometriler* course in my department at Mimar Sinan. That course met two hours a week for the 13 weeks of the fall semester of 2015–6. I kept a record like the present one. Class at the Math Village met two hours a day the first week, an hour and a half the second, Monday through Sunday, except Thursday. In the second week, I ended on Friday, so that I could participate in the Thales Buluşması in Milet the next day.

I thank the staff of the Village, notably Aslı Can Korkmaz, for having Pappus and Lobachevski printed and coil-bound for distribution to students. I thank all of the workers of the Village, from Ali Nesin on down, for making it an excellent place to continue and develop the thoughts begun more than 2500 years ago, elsewhere in Ionia, by Thales.

Contents

List of Figures

Introduction

Overview

Of Pappus's lemmas for Euclid's *Porisms,* students presented six: VIII, IV, III, X, XI, and XII, in that order. Lemmas VIII, XII, and XIII are cases of what is now known as Pappus's Theorem, while Lemmas III, X, and XI are needed to prove XII and XIII. We skipped Lemma XIII in class, its proof being similar to that of XII. Lemma IV is effectively what I shall call the Quadrangle Theorem, although Coxeter gives it no name [5, **14.41,** p. 240]. There is a related theorem called Desargues's Involution Theorem by Field and Gray [12, p. 54]; Coxeter describes this as "the theorem of the quadrangular set" [5, **14.56,** p. 246].

Two students volunteered to present the first two (VIII and IV) of Pappus's lemmas above. For the next two lemmas (III and X), volunteers were not forthcoming, and so I picked two more students. When they had fulfilled their assignments, only two more students were still in class; I asked them to prepare the last two lemmas (XI and XII) for the next day. Class met at 8 A.M., a difficult time for many. Nonetheless, some absent students did return the next day.

Every presenter of a lemma came more or less prepared for the job, though sometimes needing help from the audience. Class was mostly in Turkish, except on the last day or two, when only I was speaking: with the remaining students' permission, I switched mostly to English.

In the following week, class was in the afternoon, as one returning student had begged for it to be. Most students in the second week were new. With a couple of notable exceptions, they did not prepare their presentations well. Some of them left the Village early, earlier than I did, without telling me, and having accepted assignments for the day (Friday) when they would be gone. I have doubts about how well even the remaining students understood Lobachevski's non-Euclidean conception of parallelism: they seemed to persist in their Euclidean notions.

On the first day of that second week, I reviewed the Euclidean geometry not requiring the Fifth Postulate that Lobachevski summarizes in his Theorems 1–15. This is the geometry of Propositions 1–28 of Book I of the *Elements.* There is also some solid geometry from Book XI, though I did not go into this. I do not know how much the review of Euclidean geometry meant to students who, unlike those at Mimar Sinan, had not read Euclid in the first place. I gave away the plot by describing the Poincaré half-plane model for Lobachevskian geometry; but given the quality of later student presentations, I have doubts that the model made much sense. I shall not say more about that second week, except that, if that part of the course is repeated, it should probably be coupled with a reading of Euclid; and then it would need a full week, if not two.

Details

One may refer to Pappus's Theorem more precisely as Pappus's Hexagon Theorem. It is the theorem that, if the vertices of a hexagon lie alternately on two straight lines, then the intersection points of the three pairs of opposite sides of

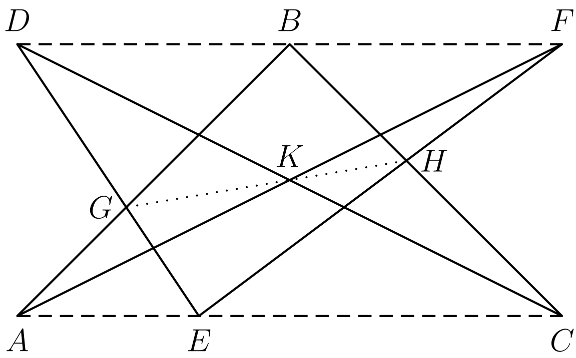


Figure 1. Pappus's Hexagon Theorem

the hexagon lie on a straight line. See for example Figure 1, which is the same as Figure 15 on page 36. Pappus's theorem remains true if some of the opposite sides of the hexagon are parallel, provided one allows that parallel lines meet on the "line at infinity"; See Figure 25 on page 47 and Figure 45 on page 70. In any case, the two straight lines holding the six vertices between them can be considered as a degenerate conic section. Pascal generalized to an arbitrary conic section, though without proof [4, 29].

The Quadrangle Theorem is that if a straight line intersects six straight lines, each of which passes through two of four given points, no three of the four being collinear, then five of the intersection points determine the sixth, regardless of the choice of the four given points. Thus in Figure 2, which is the same as Figure 12 on page 34, the points A, B, C, D, and E on the same straight line determine the point L on that line, provided it is understood that L is found by first picking a point F not on the line AB, then picking a point G on the straight line AF, letting GC and GE intersect FB and FD

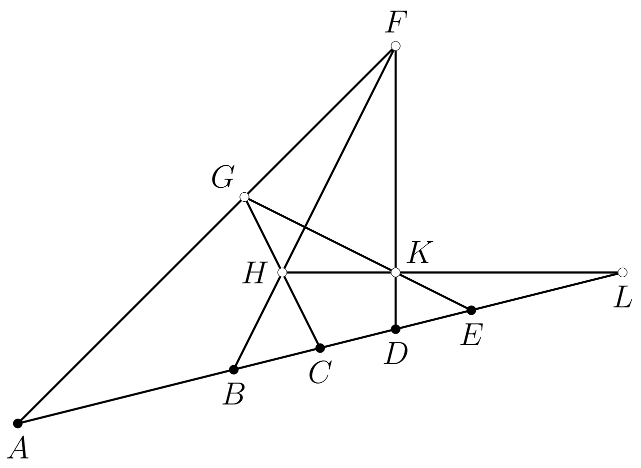


Figure 2. The Quadrangle Theorem

respectively at H and K, and letting HK intersect the original line AB.

In class, after we saw Pappus's treatment of the Quadrangle and Hexagon Theorems, I sketched a proof of all cases of the latter by passing to a third dimension and projecting. I gave another proof by introducing projective coordinates.

Desargues's Theorem is that if the straight lines through corresponding vertices of two triangles intersect at one point, then the intersection points of the corresponding sides of the triangles lie on one straight line. For example, if the triangles are ABC and DEF as in Figure 3, which is the same as Figure 14 on page 36, then HKL is straight. I gave the proof attributed to Hessenberg from 1905, using three applications of Pappus's Theorem, as sketched in Figure 53 on page 85. Strictly, this proof assumes that all points named in the diagram are distinct; in the other case, Cronheim gave in 1953 [6] the argument sketched in Figure 4, where, by applying

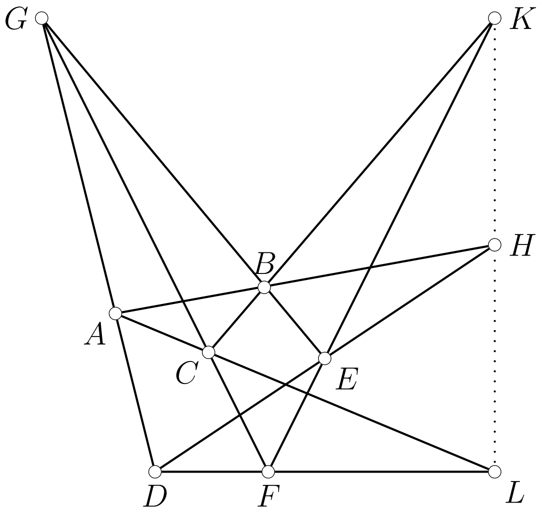


Figure 3. Desargues's Theorem

Pappus's Theorem to hexagons $GCELBA$, $GAELBC$, and $SRDCAF$ in turn, we have that FHS, DKR and finally LKH are straight.

In class, using Desargue's Theorem and its dual, which is its converse, I proved again the Quadrangle Theorem.

Pappus's proofs rely heavily on the proposition known in modern times, in some countries, as Thales's Theorem [21]: a straight line cutting two sides of a triangle cuts them proportionally if and only if it is parallel to the base. This is "equation" (12) on page 38 with respect to Figure 18. Euclid proves this theorem at the beginning of Book VI of the *Elements,* using the theory of proportion developed in Book V. This theory relies on the so-called Archimedean Axiom, that of two line segments, either can be multiplied so as to exceed the other. In fact one does not need this axiom, but one can develop a theory of proportion, sufficient for establishing

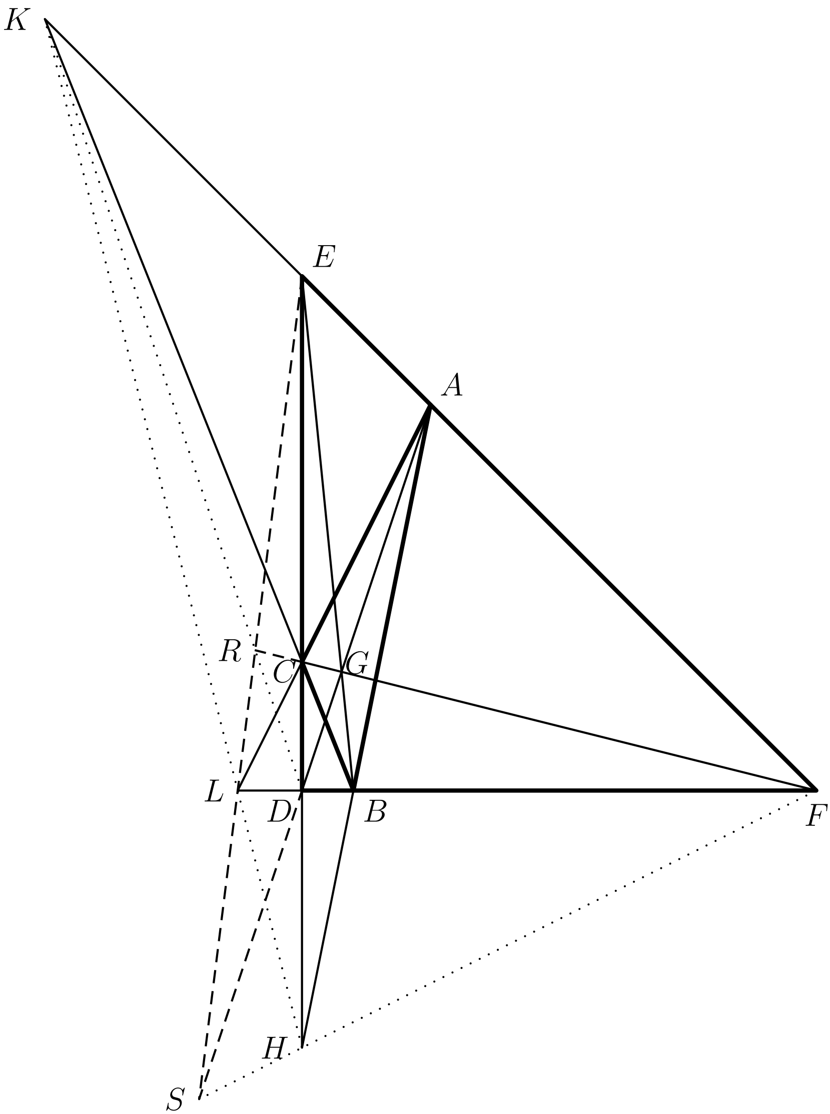


Figure 4. Degenerate case of Desargues's Theorem

Thales's Theorem, on the basis of Book I of the *Elements.* One can take Thales's Theorem as a *definition* of proportion, except that one should fix one of the base angles of the triangle. But then one can prove, as I did in class, that the base angles do not matter. A neat way of proving this is Desargues's Theorem, applied to Figure 22 on page 43. In short, Thales proves Pappus, which proves Desargues, which proves Thales.

In modern axiomatic projective plane geometry, the theorems of Pappus and Desargues are not equivalent. In class we proved, not exactly their equivalence with Thales's Theorem, but simply their truth in the geometry of Book I of Euclid's *Elements.* They are true in the projective plane over a commutative field; but the ancient proofs use not just the algebra of fields, but the geometry of *areas.* In Lemma VIII, for example, which is the case of the Hexagon Theorem when two pairs of opposite sides are parallel, Pappus's proof relies on adding and subtracting triangles that are equal because they are on the same base and between the same parallels.

Following first Descartes [9] and then Hilbert [14], we can obtain a field from Euclid's geometry. This may not be the best way to think about that geometry.

Origins

The origin of this course is my interest in the origins of mathematics. This interest goes back at least to a tenth-grade geometry class in 1980–1, where we students were taught to write proofs in the two-column, statement–reason format. I did not much care for our textbook, which, for an example of congruence, used a photo of a machine stamping out foil trays for TV dinners [31, p. 13]. In "Commensurability and Symmetry"

[25], I mention how the Weeks–Adkins text confuses equality with sameness. A geometrical equation like $AB = CD$ means not that the segments AB and CD are the same, but that their *lengths* are the same. Length is an abstraction from a segment, as ratio is an abstraction from two segments. This is why Euclid uses "equal" to describe two equal segments, but "same" to describe the ratios of segments in a proportion. One can maintain the distinction symbolically by writing a proportion as $A : B :: C : D$, rather than as $A/B = C/D$. I noticed the distinction many years after high school; but even in tenth grade I thought we should read Euclid. I went on to read him, along with Homer and Aeschylus and Plato and others, at St John's College [23].

In 2008, the first course I taught at the Nesin Mathematics Village was an opportunity to review something I had read at St John's. Called "Conic Sections *à la* Apollonius of Perga," my course reviewed the propositions of Book I of the *Conics* [1, 2] that pertained to the parabola. I shall say more about this later; for now I shall note that, while the course was great for me, I don't think it meant much for the students who just sat and watched me at the board. One has to engage with the mathematics for oneself, especially when it is something so unusual for today as Apollonius. A good way to do this is to have to go to the board and present the mathematics, as at St John's.

In 2010 at METU in Ankara, I taught the course called History of Mathematical Concepts in the manner of St John's. We studied Euclid, Apollonius, and (briefly) Archimedes in the first semester; Al-Khwārizmī, Thābit ibn Qurra, Omar Khayyám, Cardano, Viète, Descartes, and Newton in the second [22].

At METU I loved the content of the course called Fundamentals of Mathematics, which was required of all first-year students. I even wrote a text for the course, a rigorous text that might overwhelm students, but whose contents I thought at least teachers should know. In the end I didn't think it was right to try to teach equivalence-relations and proofs to beginning students, independently from a course of real mathematics. Moving to Mimar Sinan in 2011, I was able (in collaboration) to develop a course in which first-year students read and presented the proofs that taught mathematics to practically all mathematicians until about a century ago. Among other things, students would learn the non-trivial (because non-identical) equivalence-relation of congruence. I did not actually recognize this opportunity until I had seen the way students tended to confuse equality of line-segments with sameness.

Our first-semester Euclid course is followed by an analytic geometry course. Pondering the transition from the one course to the other led to some of the ideas about ratio and proportion that are worked out in the present course. My study of Pappus's Theorem arose in this context, and I was disappointed to find that the *Wikipedia* article called "Pappus's Hexagon Theorem" did not provide a precise reference to its namesake. I rectified this condition on May 13, 2013, when I added to the article a section called "Origins," giving Pappus's proof.

In order to track down that proof, I had relied on Heath, who in *A History of Greek Mathematics* summarizes most of Pappus's lemmas for Euclid's lost *Porisms* [13, p. 419–24]. In this summary, Heath *may* give the serial numbers of the lemmas as such: these are the numbers given above as Roman numerals. Heath *always* gives the numbers of the lemmas as propositions within Book VII of Pappus's *Collection,* according to the enumeration of Hultsch [18]. Apparently this enumeration was made originally in the 16th century by Commandino in his Latin translation [19, pp. 62–3, 77].[1]

According to Heath, Pappus's Lemmas XII, XIII, XV, and XVII for the *Porisms,* or Propositions 138, 139, 141, and 143 of Book VII, establish the Hexagon Theorem. The latter two propositions can be considered as converses of the former two, which consider the hexagon lying respectively between parallel and intersecting straight lines.

In *Mathematical Thought from Ancient to Modern Times,* Kline cites only Proposition 139 as giving Pappus's Theorem [16, p. 128]. This proposition, Lemma XIII, follows from Lemmas III and X, as XII follows from XI and X. For Pappus's Theorem in the most general sense, one should cite also Proposition 134, Lemma VIII, which, as we have noted, is the case where two pairs of opposite sides of the hexagon are parallel: the conclusion is then that the third pair are also parallel. Heath's summary does not seem to mention this lemma at all. The omission must be a simple oversight.

In the catalogue of my home department at Mimar Sinan, there is an elective course called Geometries, meeting two hours a week. The course had last been taught in the fall of 2010 when I offered it in the fall of 2015. For use in the course, I translated the first 19 of Pappus's 38 lemmas for Eu-

[1] Searching for Commandino's name in Jones's book reveals an interesting tidbit on page 4: Book III of the *Collection* is addressed to an otherwise-unknown teacher of mathematics called Pandrosion. She must be a woman, since she is given the feminine form of the adjective κράτιστος, -η, -ον ("most excellent"); but "in Commandino's Latin translation her name vanishes, leaving the absurdity of the polite epithet κρατίστη being treated as a name, 'Cratiste'; while for no good reason Hultsch alters the text to make the name masculine."

clid's *Porisms.* I had not thought there was an English version; but at the end of my work, I found Jones's. This helped me to parse a few confusing words. What I found first, on *Library Genesis,* was the first volume of Jones's work [19]; Professor Jones himself supplied me with Volume II, the one with the commentary and diagrams [20].

Book VII of Pappus's *Collection* is an account of the so-called *Treasury of Analysis* (ἀναλυόμενος τόπος). This *Treasury* consisted of works by Euclid, Apollonius, Aristaeus, and Eratosthenes, most of them now lost. As a reminder of the wealth of knowledge that is no longer ours, I ultimately wrote out, on the back of my Pappus translation, a table of the contents of the *Treasury.* Pappus's list of the contents is included by Thomas in his *Selections Illustrating the History of Greek Mathematics* in the Loeb series [30, pp. 598–601]. Thomas's anthology includes more selections from Pappus's *Collection,* but none involving the Hexagon Theorem. He does provide Proposition 130 of Book VII, that is, Lemma IV for the *Porisms* of Euclid, the lemma that I am calling the Quadrangle Theorem.

Additions

Here are some further developments of topics of the course.

Involutions

Lemma IV is that if the points $ABCDEL$ in Figure 2 above (or in Figure 12 on page 34) satisfy the proportion

$$AL \cdot BC : AB \cdot CL :: AL \cdot DE : AD \cdot EL, \tag{1}$$

then HKL is straight. Chasles [3, p. 102] rewrites the condition in the form

$$BC \cdot AD \cdot EL = AB \cdot DE \cdot CL.$$

We can also write

$$BC \cdot AD : AB \cdot DE :: CL : EL;$$

this makes it easier to see that, when $ABCDE$ are given, then some unique L exists so as to satisfy (1). As L varies, the ratio $CL : EL$ takes on all possible values but unity. If $BC \cdot AD = AB \cdot DE$, that is,

$$AB : BC :: AD : DE,$$

then $HK \parallel AE$ by Lemma I. Otherwise, by Lemma IV, HK must pass through the L that satisfies (1). Thus the converse of the lemma holds as well. We might speculate whether this converse was one of Euclid's original porisms. Chasles seems to think it was.

Thomas observes [30, pp. 612–3] that

> [the converse of Lemma IV] is one of the ways of expressing the proposition enunciated by Desargues: *The three pairs of opposite sides of a complete quadrilateral are cut by any transversal in three pairs of conjugate points of an involution.*

Following Coxeter, I would call the "complete quadrilateral" here a *complete quadrangle* (see Figure 13, page 35). Desargues proves the **Involution Theorem** in his *Rough Draft of an Essay on the results of taking plane sections of a cone* [12, p. 54]. One way to interpret the theorem is to observe that, in Figure 2 or 12, if the points $BCDE$ are conceived of as fixed,

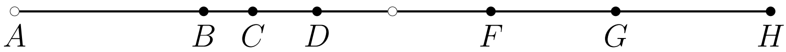


Figure 5. Points in involution

then A determines L. Moreover, this operation *transposes* A and L, and so it is an *involution* of the straight line BE.

Desargues proceeds towards the Involution Theorem by first observing that if seven points $ABCDFGH$ are arranged, as in Figure 5, on a straight line so that

$$AB \cdot AH = AC \cdot AG = AD \cdot AF, \tag{2}$$

then, without reference to A, we have [12, p. 48]

$$DG \cdot FG : CD \cdot CF :: BG \cdot GH : BC \cdot CH. \tag{3}$$

Indeed, by second equation in (2),

$$AG : AF :: AD : AC, \tag{4}$$

so by addition or subtraction of the terms of the first ratio to or from the terms of the second,

$$AG : AF :: DG : CF. \tag{5}$$

Similarly, after alternating (4), we obtain

$$AF : AC :: AG : AD,$$
$$AF : AC :: FG : CD.$$

Composing the latter with (5) yields

$$AG : AC :: DG \cdot FG : CD \cdot CF. \tag{6}$$

Replacing D with B and F with H in the second equation of (2) yields the first equation. Hence we can do the same in (6), obtaining

$$AG : AC :: BG \cdot GH : BC \cdot CH.$$

Eliminating the common ratio from the last two proportions yields (3). For meeting this condition, the three pairs BH, CG and DF of points are said to be in **involution,** by Desargues's definition.

Now suppose these pairs consist of those points where a transversal cuts the pairs of opposite sides of a complete quadrangle, as in Thomas's description. For example, the points $BCDFGH$ could be, respectively, the points $ABCDEL$ in Figure 2 or 12, or ΑΒΓΔΕΖ in Figure 26 on page 48. Then the converse of Pappus's Lemma IV, expressed in (1), gives us now

$$BH \cdot CD : BC \cdot DH :: BH \cdot FG : BF \cdot GH. \tag{7}$$

This proportion is equivalent to (3). However, the best way to show this may not be obvious. One approach is to introduce the **cross-ratio,** though apparently Desargues does not do this [12, p. 52]. Despite the misgivings expressed above (page 15), we turn to modern notation for ratios. If $ABCD$ are points on a straight line, we let

$$\frac{AB \cdot CD}{AD \cdot CB} = (ABCD) \tag{8}$$

by definition. Note the pattern of repeated letters on the left. We could use a different pattern; we just have to be consistent. We take line segments to be directed, so that $CB = -BC$. Then (7) is equivalent to

$$(BHDC) = (BHGF), \tag{9}$$

while, since (3) is, in modern notation,

$$\frac{DG \cdot FG}{CD \cdot CF} = \frac{BG \cdot GH}{BC \cdot CH},$$

we obtain from this

$$\frac{DG \cdot BC}{CD \cdot BG} = \frac{CF \cdot GH}{FG \cdot CH},$$

that is,

$$(DGBC) = (CFGH). \tag{10}$$

But (9), obtained from Pappus, must still hold if we permute the pairs BH, CG, and DF. Sending each pair to the next (and the last to the first), we obtain from (9) the equivalent equation

$$(CGBD) = (CGFH). \tag{11}$$

We shall have that this is equivalent to (10), once we understand how cross-ratios are affected by permutations of their entries.

The cross-ratios that can be formed from $ABCD$ are permuted transitively by a group of order 24. Then there are at most 6 different cross-ratios, since

$$(CDAB) = (ABCD),$$
$$(BADC) = (ABCD).$$

Moreover, from (8) we can read off

$$(ADCB) = \frac{1}{(ABCD)},$$

while

$$(ACBD) = \frac{AC \cdot BD}{AD \cdot BC} = \frac{(AB + BC) \cdot (BC + CD)}{AD \cdot BC}$$
$$= \frac{AC \cdot BC + BC \cdot CD}{AD \cdot BC} - (ABCD)$$
$$= 1 - (ABCD).$$

The involutions $x \mapsto 1/x$ and $x \mapsto 1 - x$ of the set of ratios generate a group of order 6. (Here we either exclude the ratios 0 and 1, or allow them along with ∞.) Now we have accounted for all permutations of points. We have

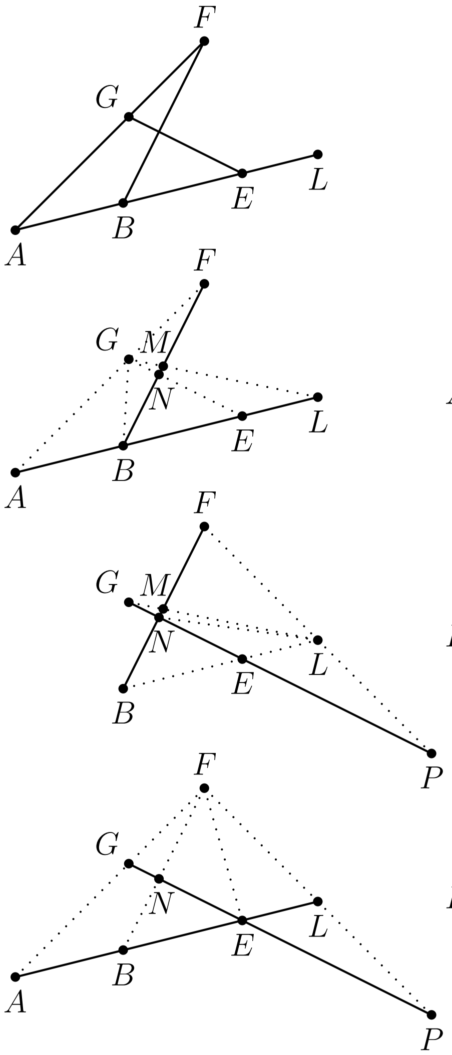
$$\begin{aligned}(CGBD) = (CGFH) &\iff (CBGD) = (CFGH)\\ &\iff (BCDG) = (CFGH)\\ &\iff (DGBC) = (CFGH),\end{aligned}$$

that is, (11) and (10) are equivalent, as desired.

The involution of the straight line BE in Figure 2 or 12 that transposes A and L will transpose B and E, and also C and D. Following Coxeter [5, **14.56**, p. 246], we obtain this transposition as in Figure 6, as a composition of three projections of one straight line onto another. In Coxeter's notation, $ABEL \overset{G}{\underset{\wedge}{=}} FBNM$ means AGF, EGN, and LGM are all straight (and implicitly $ABEL$ and $FBNM$ are straight). It follows from Lemma IV that this transformation is an involution and is uniquely determined by the pairs AL and BE.

Locus problems

Thomas's anthology [30, 600–3] includes the account by Pappus of five- and six-line locus problems that Descartes quotes



$$ABEL \underset{\wedge}{\overset{G}{=}} FBNM$$

$$FBNM \underset{\wedge}{\overset{L}{=}} PENG$$

$$PENG \underset{\wedge}{\overset{F}{=}} LEBA$$

Figure 6. An involution

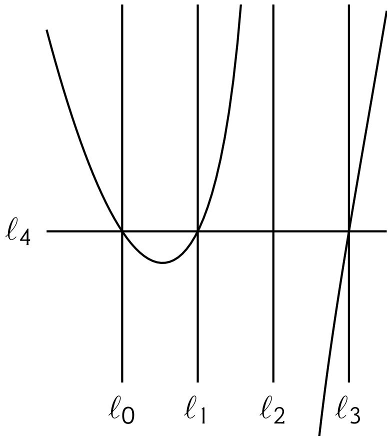


Figure 7. A five-line locus

in the *Geometry* [7, pp. 18–21]. Pappus suggests no solution to such problems; but later in the *Geometry,* Descartes solves a special case of the five-line problem, where four of the straight lines—say ℓ_0, ℓ_1, ℓ_2, and ℓ_3—are parallel to one another, each a distance a from the previous, while the fifth line—ℓ_4—is perpendicular to them. What is the locus of points such that the product of their distances to ℓ_0, ℓ_1, and ℓ_3 is equal to the product of a with the distances to ℓ_2 and ℓ_4? One can write down an *equation* for the locus, and Descartes does. Effectively letting the x- and y-axes be ℓ_2 and ℓ_4, the positive direction of the latter being from ℓ_3 towards ℓ_0, Descartes obtains

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy.$$

This may allow us to plot points on the desired locus, as in Figure 7; but we could already do that. The equation is thus not a solution to the locus problem, since it does not tell us what the locus *is.* But Descartes shows that the locus is traced

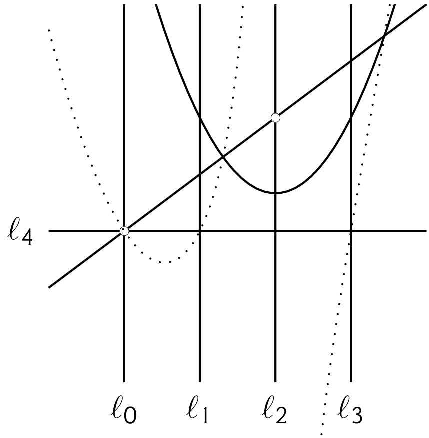


Figure 8. Solution of the five-line locus problem

by the intersection of a moving parabola with a straight line passing through one fixed point and one point that moves with the parabola. See Figure 8. The parabola has axis sliding along ℓ_2, and a is its *latus rectum.*[2] The straight line passes through the intersection of ℓ_0 and ℓ_4 and through the point on the axis of the parabola whose distance from the vertex is a.

Descartes's solution is apparently one that Pappus would recognize as such. Thus Descartes's algebraic methods would seem to represent an advance, and not just a different way of doing mathematics.

[2]See my article "Abscissas and Ordinates" [24] for more than you ever imagined wanting to know about the term *latus rectum.*

Ancients and moderns

Modern mathematics is *literal* in the sense of relying on the letters that might appear in a diagram, rather than the diagram itself. The diagram is an integral part of ancient proofs. Pappus may put points of a diagram in a list, without giving all of their relations. For example, the enunciation of Lemma I of our text is,

Ἔστω καταγραφὴ	Let the diagram be
ἡ ΑΒΓΔΕΖΗ,	ΑΒΓΔΕΖΗ,
καὶ ἔστω	and let it be that
ὡς ἡ ΑΖ πρὸς τὴν ΖΗ,	as ΑΖ is to ΖΗ,
οὕτως ἡ ΑΔ πρὸς τὴν ΔΓ,	so is ΑΔ to ΔΓ;
καὶ ἐπεζεύχθω ἡ ΘΚ·	and let ΘΚ have been joined;
ὅτι	[I say] that
παράλληλός ἐστιν ἡ ΘΚ τῇ ΑΓ.	ΘΚ is parallel to ΑΓ.

Reconstruct the diagram from that! One needs to know, first of all, that ΑΖΗΔΓ are on a straight line. ΑΒΕ are also on a straight line, and one needs to know that Λ is to be on this line when ΖΛ is constructed parallel to ΒΔ. We can infer the locations of Θ and Κ from Pappus's assertion that, as ΕΒ is to ΒΛ, so are ΕΚ to ΚΖ and ΕΘ to ΘΗ; but still we must understand that each ratio involves collinear points. In his diagrams, Hultsch gives five possible configurations meeting these conditions; my translation gives a sixth, as in Figure 9, where ΒΕΚΘ is emboldened to indicate that the lemma is a version of the Quadrangle Theorem where one of the six straight lines through the vertices of the quadrangle is parallel to a straight line cutting the others.

Netz observes that Greek mathematics *never* simply declares what letters in a diagram stand for [17, pp. 24–5]:

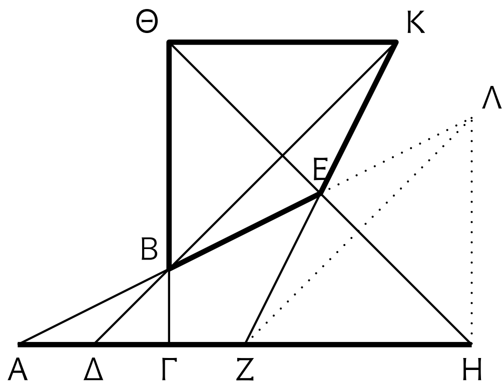


Figure 9. A diagram for Lemma I of Pappus

> Nowhere in Greek mathematics do we find a moment of specification *per se,* a moment whose purpose is to make sure that the attribution of letters in the diagram is fixed.

It may well be Descartes, he says, who first fixes such attributions. I note an early passage in *La Géométrie* [9, p. 2]:

> Mais souvent on n'a pas besoin de tracer ainsi ces lignes sur le papier, et il suffit de les désigner par quelques lettres, chacune par une seule. Comme pour ajouter la ligne BD à GH, je nomme l'une a et l'autre b, et écris $a + b$; et $a - b$ pour soustraire b de a; et ab pour les multiplier l'une par l'autre . . .

We have already seen how Descartes's abbreviations make solutions to ancient unsolved problems possible.

Nonetheless, it it possible to take cleverness with notation too far. The third proof of Pappus's Theorem given in the course is taken from Coxeter [5, p. 236], and it may appeal to modern Cartesian sensibilities; but it gives less sense of *why* the theorem is true than the second proof, which involves

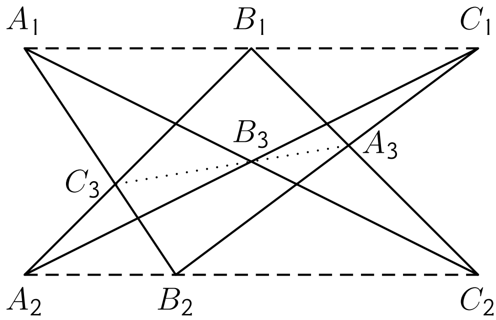


Figure 10. Pappus's Theorem in modern notation

passing to a third dimension and *projecting,* starting from the diagram of Lemma VIII. Coxeter's diagram for Pappus's Theorem is labelled as in Figure 10; it allows us to observe, once for all, that $A_iB_jC_k$ is straight whenever $\{i, j, k\} = \{1, 2, 3\}$. If one is going to be this clever, it seems to me, one might as well go all the way and let the indices be 0, 1, and 2, so that one can use for the set of them the notation 3.

I do not know whether Pappus (or Euclid) recognized a single theorem lying behind Lemmas VIII, XII, and XIII. He may have recognized a similarity between the lemmas, but not a satisfactory way to prove the lemmas once for all. He may have thought it important to treat each case individually.

There is a point of view that blurs the distinctions between the three conic sections; but it not necessarily the best point of view. When Omar Khayyām used conic sections to solve what we should call cubic equations, he had to consider several cases, depending on what we should call the signs of the coefficients of the equations [15, pp. 556–60]. For example, for the case where "a cube and sides are equal to squares and numbers," we can write the problem as the equation

$$x^3 + b^2x = cx^2 + b^2d,$$

which we manipulate into

$$\frac{x^2}{b^2} = \frac{d-x}{x-c}.$$

We can define y in terms of a solution so that

$$\frac{x}{b} = \frac{y}{x-c} = \frac{d-x}{y}.$$

Thus we solve the original equation by finding the intersection of the two conics given by

$$x^2 - cx = by, \qquad y^2 + (x-c)\cdot(x-d) = 0.$$

These are normally a parabola and a circle; however, if we have allowed negative coefficients, then we may have had to let b be imaginary. This matters if we want to construct real solutions.

In Rule Four of the posthumously published *Rules for the Direction of the Mind* [8, 373, p. 17], Descartes writes of a method that is

> so useful . . . that without it the pursuit of learning would, I think, be more harmful than profitable. Hence I can readily believe that the great minds of the past were to some extent aware of it, guided to it even by nature alone . . . This is our experience in the simplest of sciences, arithmetic and geometry: we are well aware that the geometers of antiquity employed a sort of analysis which they went on to apply to the solution of every problem, though they begrudged revealing it to posterity. At the present time a sort of arithmetic called "algebra" is flourishing, and this is achieving for numbers what the ancients did for figures.

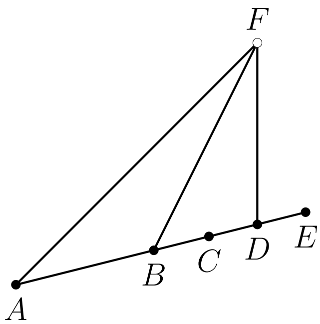
We have already observed that Book VII of Pappus's *Collection* concerned the *Treasury of Analysis.* The term "treasury" is a modern flourish; but our word "analysis" is a transliteration of the Greek of writers like Pappus. It means freeing up, or dissolving. As Pappus describes it, mathematical analysis is assuming what you are trying to find, so that you can work backwards to see how to get there. We do this today by giving what we want to find a name x.

Today we think of conic sections as having *axes*: one for the parabola, and to each for the ellipse and hyperbola. The notion comes from Apollonius; but for him, an axis is just a special case of a *diameter.* A diameter of a conic section bisects the chords of the section that are parallel to a certain straight line. This straight line is the tangent drawn at a point where the diameter meets the section. Apollonius shows that *every* straight line through the center of an ellipse or hyperbola is a diameter in this sense; and every straight line parallel to the axis is a diameter of a parabola. As far as I can tell, it is difficult to show this by the method's of Descartes's analytic geometry. Like Euclid's and Pappus's proofs, Apollonius's proofs rely on *areas.* There are areas of scalene triangles and non-rectangular parallelograms. Earlier I discussed a locus problem in terms of *distances* from several given straight lines. For Pappus, what is involved is not distances as such, but the lengths of segments drawn to the given lines at given angles, which are not necessarily right angles. The apparently greater generality is trivial. This is why Descartes can solve a five-line problem using algebra. But if one is going to prove that a straight line parallel to the axis of a parabola is a diameter, one cannot just treat all angles as right. Apollonius may have had a secret weapon in coming up with his propositions about conic sections; but I don't think it was Cartesian analysis. Such at least was my impression when going through Apollonius for my course in Şirince, in 2008.

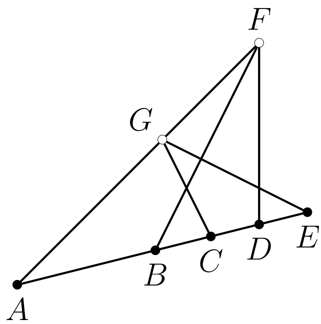
Monday, September 12, 2016

Statement of the Quadrangle Theorem

Suppose five points, A through E, fall on a straight line as in Figure 11a, and F is a random point not on the straight line. Join FA, FB, and FD. Now let G be a random point on FA, as in Figure 11b, and join GC and GE. Supposing these two straight lines cross FB and FD at H and K respectively, join HK as in Figure 12. If this straight line crosses the original straight line AB at L, then L depends only on the original five points, not on F or G. Let us call this result the **Quadrangle Theorem.** It is about how the straight line AB crosses the six straight lines that pass through pairs of the four points F, G, H, and K. Any such collection of four points, no three of which are collinear, together with the six straight lines that they



(a) Point F is chosen



(b) Point G is chosen

Figure 11. The Quadrangle Theorem set up

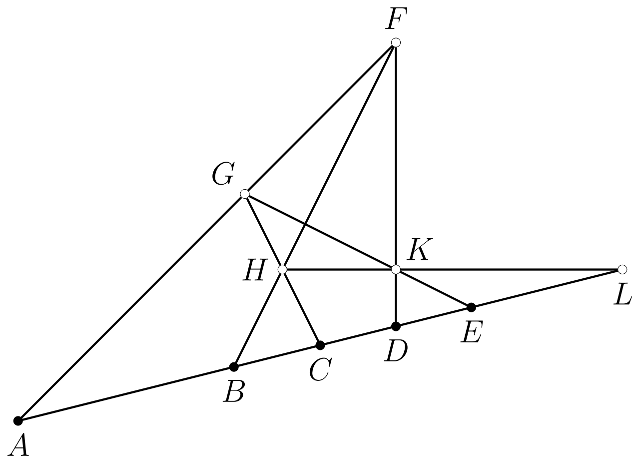


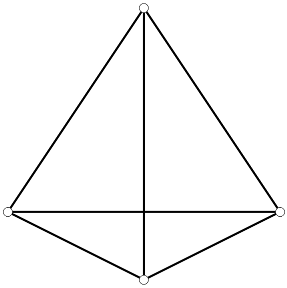
Figure 12. The Quadrangle Theorem

determine, as in Figure 13a, is called a **complete quadrangle** (*tam dörtgen*). Similarly, any collection of four straight lines, no three passing through the same point, together with the six points at the intersections of pairs of these six straight lines, as in Figure 13b, is a **complete quadrilateral** (*tam dörtkenar*).

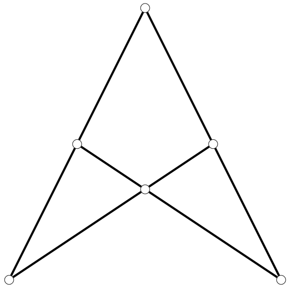
As stated, the Quadrangle Theorem is a consequence of Lemma IV of our text of Pappus. Lemmas I, II, V, VI, and VII treat other cases, such as when HK in Figure 12 is parallel to AE.

Statement of Desargues's Theorem

There is an alternative proof of the Quadrangle Theorem based on Desargues's Theorem. Desargues was a contemporary of Descartes, and the theorem named for him is that, if the straight lines through corresponding vertices of two triangles



(a) A complete quadrangle



(b) A complete quadrilateral

Figure 13. "Complete" figures

intersect at a common point, as AD, BE, and CF meet at H in Figure 14, then the intersection points of corresponding sides of the triangles lie in a single straight line. If two pairs of corresponding sides are parallel, this means the third pair must be parallel as well.

Statement of Pappus's Theorem

We shall prove Desargues's Theorem by means of **Pappus's Hexagon Theorem.** This is the theorem that if the vertices of a hexagon lie alternately on two straight lines, as do the vertices of $ABCDEF$ in Figure 15, then the points of intersection, as G, H, and K, of the three pairs of opposite sides of the hexagon also lie on a straight line. Again there are other cases, as when two of the pairs of opposite sides are parallel; here the third pair must be parallel as well.

Pappus's Theorem is Lemmas VIII, XII and XIII in our text. In the proofs of XII and XIII, Lemmas III, X, and XI are used. Lemma VIII needs only the theorem of Euclid (namely I.37

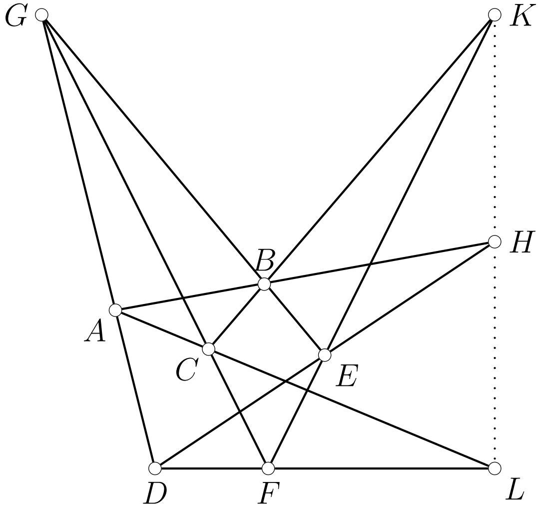


Figure 14. Desargues's Theorem

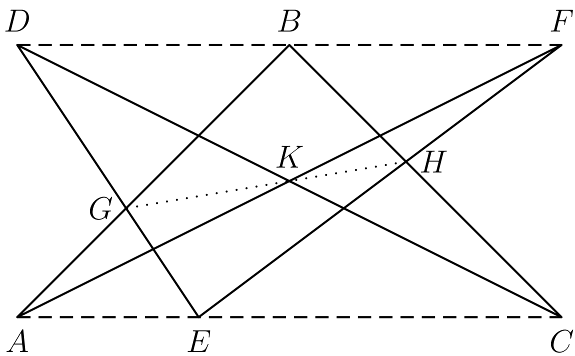


Figure 15. Pappus's Hexagon Theorem

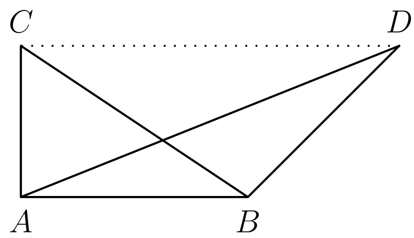


Figure 16. Triangles on the same base

and 39 of the *Elements* [11]) that when two triangles have the same base, as in Figure 16, then the triangles are equal to one another just in case the straight line joining the apices of the triangles is parallel to their common base.

Equality and sameness

Equality of triangles means not congruence of the triangles, but sameness of their areas. In fact Euclid's proof that *parallelograms* on the same base and in the same parallels are equal is by cutting the parallelograms into congruent pieces, as in Figure 17, where parallelograms $ABDC$ and $ABFE$ are equal, because, when $CH = DG$ and $FL = EG$, while CK and FM are both equal to DE, then triangles CHK and MLF are congruent to one another (and to DGE), and the pentagons $AGDKH$ and $BLMEG$ are congruent to one another. Being half of equal parallelograms, triangles ABC and ABE are now equal to one another.

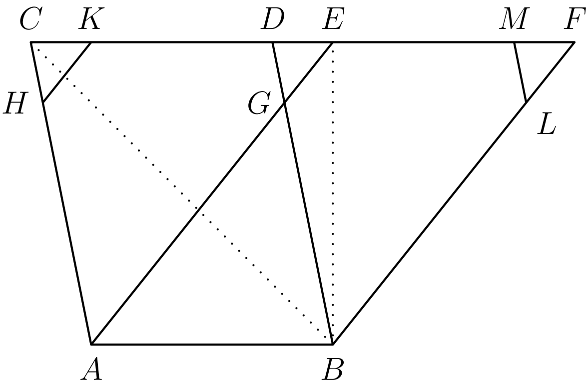


Figure 17. Equal parallelograms

Statement of Thales's Theorem

After Lemma VIII, Pappus's proofs of other cases of the Hexagon Theorem use what is known as **Thales's Theorem.** This is that if the straight line DE cuts the sides of triangle ABC as in Figure 18, then

$$DE \parallel BC \iff AD : DB :: AE : EC. \tag{12}$$

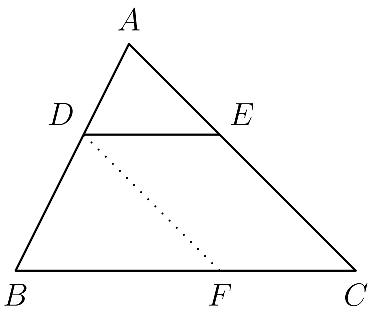


Figure 18. Thales's Theorem

We obtain the alternative formulation

$$DE \parallel BC \iff AD : AB :: DE : BC$$

by drawing DF parallel to AC and using a rule like

$$a : b :: c : d \iff a + b : b :: c + d : d$$

The proportion $AD : DB :: AE : EC$ is an expression not of the *equality* of the ratios $AD : DB$ and $AE : EC$, but of their *sameness*. An equation of bounded straight lines, like $CH = DG$ earlier, says that the *lengths* of the straight lines are the same. But the length of a line cannot be drawn in a diagram; it is an abstraction from the line itself. Equality of bounded straight lines is an equivalence relation, and for the sake of having a formal definition, we can understand the length of CH as the corresponding equivalence class, consisting of all of the straight lines that are equal to CH. Similarly, the ratio of two straight lines is abstract and cannot be drawn in a diagram. We shall discuss proportions later today and more formally on Wednesday.

Thales himself

There is little evidence that Thales knew, in full generality, the theorem named for him. I have learned this while preparing for the Thales Meeting to be held on Saturday, September 24, in Thales's home town of Miletus. Thales supposedly measured the heights of the Pyramids by considering their shadows; but he may have done this just when his own shadow was as long as a person is tall, since in this case the height of the pyramid would be the same as the length of its own shadow (as measured from the center of the base of course).

585 B.C.E.	Thales
300	Euclid
250 C.E.	Diogenes Laërtius
340	Pappus
450	Proclus

Figure 19. Dates of some ancient writers and thinkers

Thales may have recognized that two triangles are congruent if they have two angles equal respectively to two angles and the common sides equal. (This is the so-called Angle-Side-Angle or ASA Theorem.) According to Proclus, who wrote a commentary on the Book I of Euclid's *Elements* [26], Thales also knew the following three theorems found in that book:

1) the diameter of a circle divides the circle into two equal parts,
2) vertical angles formed by intersecting straight lines are equal to one another;
3) the base angles of an isosceles triangle are equal to one another.

According to Diogenes Laërtius [10, i.24–5] (who wrote biographies of philosophers, starting with Thales), Thales also knew that

4) the angle inscribed in a semicircle is right.

Dates of activity for the persons we have named are roughly as in Figure 19. All four of the listed theorems can be understood to be true by *symmetry.* For example, the equation

$$\angle ABC = \angle CBA$$

basically establishes the equality of vertical angles. Also, suppose we complete the diagram of an angle inscribed in a semicircle as in Figure 20. Here the quadrilateral $BCDE$ has four

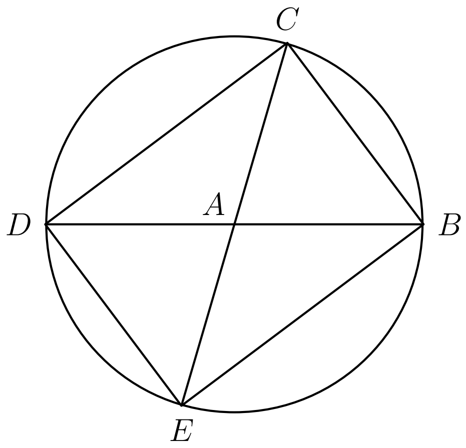


Figure 20. The angle in a semicircle

equal angles. If it follows that those angles must be right, then the theorem of the semicircle is proved.

Those four equal angles are right in Euclidean geometry. Here, by Euclid's fifth postulate, if the angles at DCB and CBE are together less then two right angles, then CD and BE must intersect when extended. In that case, for the same reason, they intersect when extended in the other direction; but this would be absurd.

Non-Euclidean planes

In Lobachevski's geometry, Euclid's fifth postulate is denied. In this case, if AB is perpendicular to AC as in Figure 21, there will be an angle ABD, less than a right angle, such that BD never intersects AC, no matter how far extended, though straight lines through B making with BA an angle less than

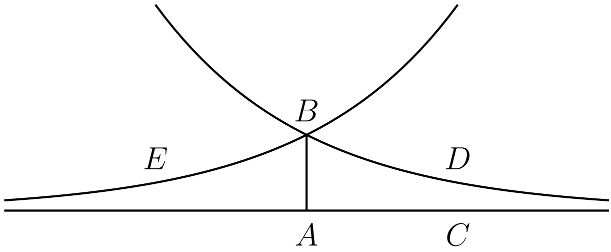


Figure 21. Straight lines in the hyperbolic plane

ABD will meet AC. If $EBA = ABD$, then BE too will never meet AC, though BE and BD are different straight lines.

Lobachevski's geometry may be understood as taking place in the **hyperbolic plane.** We can understand Pappus's geometry as taking place in the **projective plane.** Here the several cases of (for example) the Hexagon Theorem can be given a single expression, because formerly parallel straight lines are now allowed to intersect "at infinity." We obtain the projective plane from the Euclidean plane by adding,

1) for each family of parallel straight lines, a new point common to all of them, namely their **point at infinity,** and
2) a new straight line, the **straight line at infinity,** which consists precisely of all of the points at infinity.

In the projective plane,

1) any two distinct points lie on a single straight line, as in Euclidean geometry; but now also
2) any two distinct straight lines intersect at a single point (which could be at infinity).

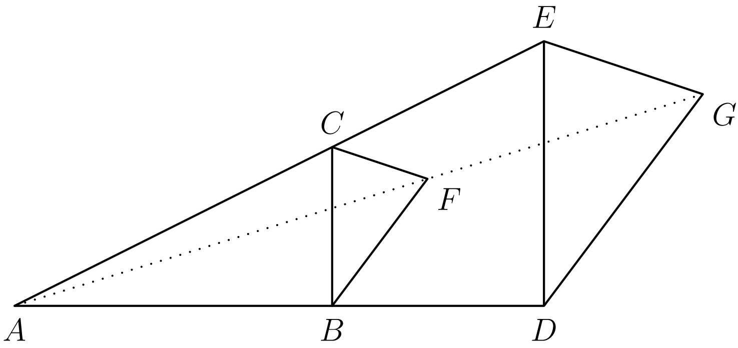


Figure 22. For a definition of proportion

Proportions

We said that Pappus's proof of the Hexagon Theorem used Thales's Theorem. How can we prove Thales's Theorem? We first have to define proportions. One approach is to say that, by definition, in Figure 22, where angles ABC and ADE are right, we have

$$AB : BC :: AD : DE.$$

But then, if $BF = BC$ and $DG = DE$, and BF is parallel to DG, we want to show that AFG is a straight line. This is a consequence of Desargues's Theorem. Indeed, in place of the assumption that $DG = DE$, assume that AFG is straight. Since $BC \parallel DE$ and $BF \parallel DG$, it follows by Desargues that $CF \parallel EG$. In that case, since in triangle BCF the angles at C and F are equal to one another, the same is true of the angles at E and G in triangle DEG. (This uses the consequence of Euclid's fifth postulate that parallel straight lines make equal angles with the same straight line.) As a result, $DE = DG$. Thus, if this equation is true by construction, then AFG must be straight.

$$
\begin{tikzcd}
 & \text{Desargues} \arrow[dd, Rightarrow] \\
\text{Pappus} \arrow[ur, Rightarrow] & \\
 & \text{Thales} \arrow[ul, Rightarrow]
\end{tikzcd}
$$

Figure 23. A circle of implications

In the end, we shall have the implications in Figure 23, at least with the assumption of a certain part of Book I of Euclid's *Elements*.

In fact the theorems of Desargues, Pappus, and Thales are all simply true in the full geometry of Book I of Euclid's *Elements*. We shall show this for Thales's Theorem on Wednesday.

Tuesday, September 13, 2016

The parallel case of Pappus's Theorem

We can understand **Pappus's Theorem** (or the Hexagon Theorem) as having the six cases depicted in Figure 24. In each case, the vertices of hexagon $ABCDEF$ lie alternately on two straight lines; but these may be parallel or not, and of the pairs (AB, DE) and (BC, EF) of opposite sides of the hexagon, two, one, or none may be parallel.

Pappus's **Lemma VIII** is the case where the two straight lines intersect, and the two pairs of opposite sides of the hexagon are parallel. In particular, letting the hexagon be $\mathsf{B\Gamma HE\Delta Z}$ in Figure 25a, we suppose

$$\mathsf{B\Gamma} \parallel \mathsf{\Delta E}, \qquad \mathsf{HE} \parallel \mathsf{ZB}.$$

We prove $\mathsf{\Gamma H} \parallel \mathsf{\Delta Z}$ using several equations of triangles:

1)	$\mathsf{\Delta BE} = \mathsf{\Delta\Gamma E}$,	[*Elements* I.37, since $\mathsf{B\Gamma} \parallel \mathsf{\Delta E}$]
2)	$\mathsf{ABE} = \mathsf{\Gamma\Delta A}$,	[add $\mathsf{\Delta AE}$]
3)	$\mathsf{BZE} = \mathsf{BZH}$,	[*Elements* I.37, since $\mathsf{BZ} \parallel \mathsf{EH}$]
4)	$\mathsf{ABE} = \mathsf{AHZ}$,	[subtract ABZ]
5)	$\mathsf{A\Gamma\Delta} = \mathsf{AHZ}$,	[steps 2 and 4]
6)	$\mathsf{\Gamma\Delta H} = \mathsf{\Gamma ZH}$,	[add $\mathsf{A\Gamma H}$]
7)	$\mathsf{\Gamma H} \parallel \mathsf{\Delta Z}$.	[*Elements* I.39]

In fact, if Pappus's figure were more like those in Figure 24, it might be as in Figure 25b; and then the proof would have to

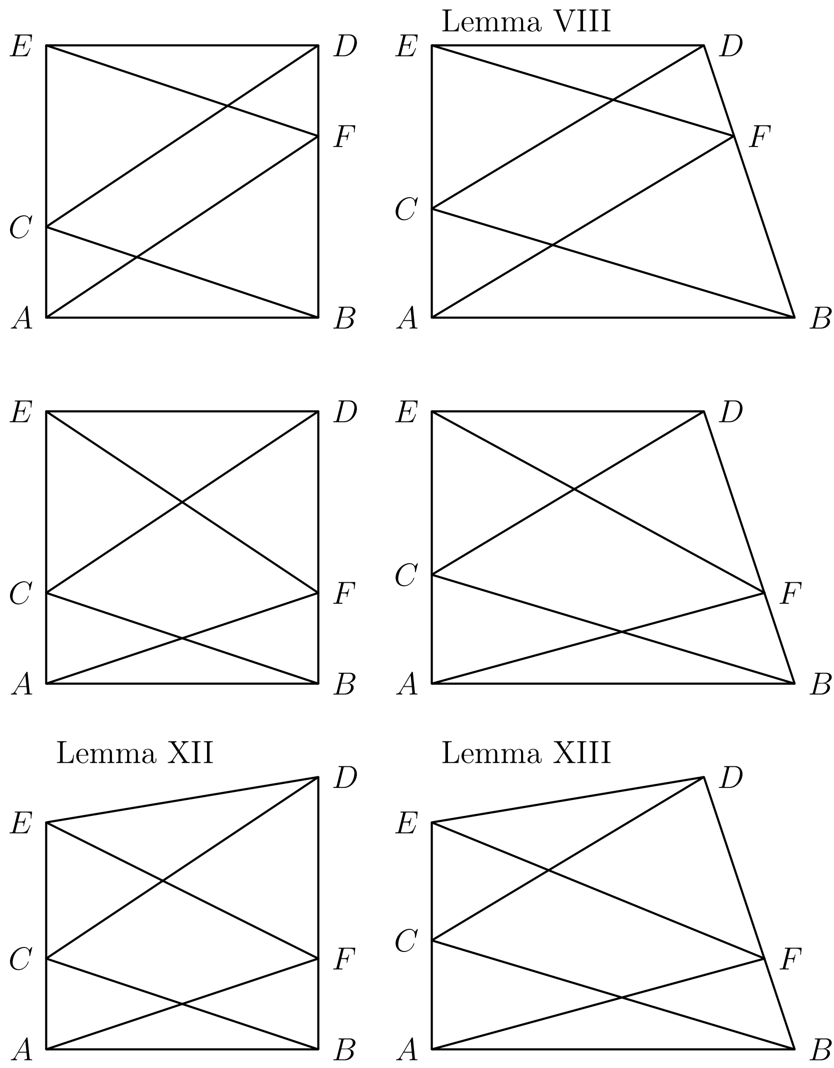
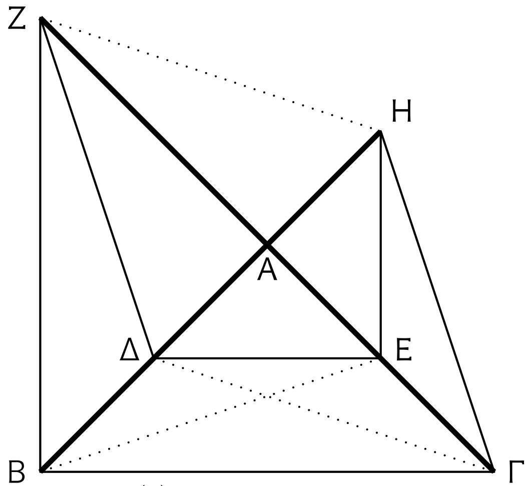
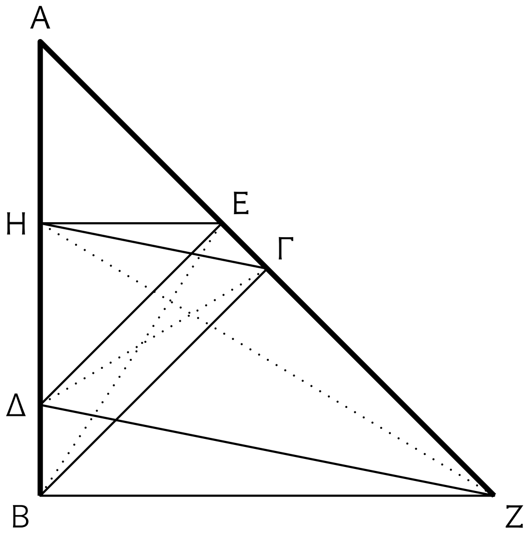


Figure 24. Cases of Pappus's Theorem



(a) Lemma VIII itself



(b) Another version of Lemma VIII

Figure 25. A case of Pappus's Theorem

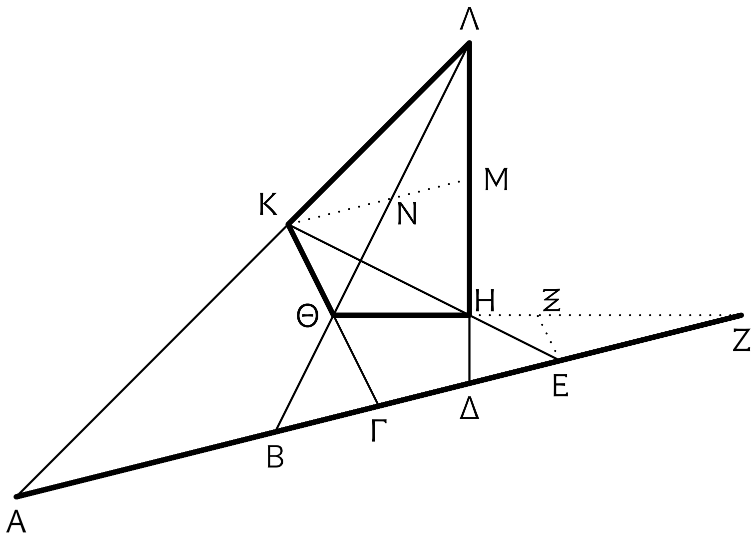


Figure 26. Lemma IV

be adjusted by, for example, subtracting BZE and BZH *from* ABZ in step 4.

Pappus's proof of the Quadrangle Theorem

Lemma IV is that if, as in Figure 26, the given points A, B, Γ, Δ, E, and Z along a straight line satisfy the proportion

$$\mathsf{AZ} \cdot \mathsf{B\Gamma} : \mathsf{AB} \cdot \mathsf{\Gamma Z} :: \mathsf{AZ} \cdot \mathsf{\Delta E} : \mathsf{A\Delta} \cdot \mathsf{EZ}, \tag{13}$$

then Θ, H, and Z are in a straight line. To prove this, by alternation of (13), we obtain

$$\mathsf{AZ} \cdot \mathsf{B\Gamma} : \mathsf{AZ} \cdot \mathsf{\Delta E} :: \mathsf{AB} \cdot \mathsf{\Gamma Z} : \mathsf{A\Delta} \cdot \mathsf{EZ}. \tag{14}$$

Considering first the left-hand member, we have by simplification

$$\mathsf{AZ} \cdot \mathsf{B\Gamma} : \mathsf{AZ} \cdot \mathsf{\Delta E} :: \mathsf{B\Gamma} : \mathsf{\Delta E},$$

and we write the latter ratio as a composition of three ratios:

$$\mathsf{B\Gamma} : \mathsf{\Delta E} :: \mathsf{B\Gamma} : \mathsf{KN} \ \ \& \ \ \mathsf{KN} : \mathsf{KM} \ \ \& \ \ \mathsf{KM} : \mathsf{\Delta E}.$$

Now we analyze the right-hand member of (14) as a composite:

$$\mathsf{AB} \cdot \mathsf{\Gamma Z} : \mathsf{A\Delta} \cdot \mathsf{EZ} :: \mathsf{BA} : \mathsf{A\Delta} \ \ \& \ \ \mathsf{\Gamma Z} : \mathsf{ZE}.$$

Assuming KM is drawn parallel to AZ, by Thales's Theorem we have

$$\mathsf{NK} : \mathsf{KM} :: \mathsf{BA} : \mathsf{A\Delta}.$$

Eliminating this common ratio from either member of (14), and reversing the order of the new members, we obtain

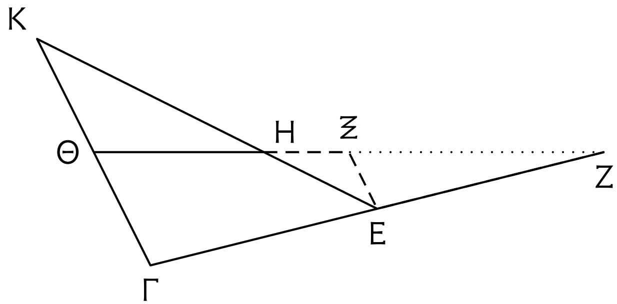
$$\mathsf{\Gamma Z} : \mathsf{ZE} :: \mathsf{B\Gamma} : \mathsf{KN} \ \ \& \ \ \mathsf{KM} : \mathsf{\Delta E},$$

and therefore, by Thales's Theorem applied to each ratio in the compound,

$$\mathsf{\Gamma Z} : \mathsf{ZE} :: \mathsf{\Theta\Gamma} : \mathsf{K\Theta} \ \ \& \ \ \mathsf{KH} : \mathsf{HE}. \tag{15}$$

Pappus says now that $\mathsf{\Theta HZ}$ is indeed straight. Although he provides a reminder, he may expect his readers to know, as some students today know from high school, the generalization of Thales's Theorem known as **Menelaus's Theorem** whose diagram is in Figure 27.[3] Here $\mathsf{E\Xi} \parallel \mathsf{\Theta\Gamma}$, and $\mathsf{\Theta H}$ is extended

[3]Menelaus's *Sphaerica* survives in Arabic translation [13, p. 261]; but we also have Menelaus's Theorem in Ptolemy, where I read it as a student at St John's College, just before Toomer's 1984 translation [28] came out;



$$\mathsf{\Gamma Z : ZE :: \Gamma\Theta : \Theta K} \;\;\&\;\; \mathsf{KH : HE} \iff \mathsf{\Theta HZ} \text{ is straight}$$

Figure 27. Lemma from Lemma IV

to Ξ. Then from (15) we have

$$\begin{aligned} \mathsf{\Gamma Z : ZE} &:: \mathsf{\Theta\Gamma : K\Theta} \;\;\&\;\; \mathsf{K\Theta : E\Xi} \\ &:: \mathsf{\Theta\Gamma : E\Xi}. \end{aligned}$$

we used the translation that Taliaferro had made for the College [27, I.13, p. 26]. Thomas also puts Menelaus's Theorem in his anthology [30, pp. 445 ff.]. In the commentary for their translation of Desargues, Field and Gray remark that Pappus's Lemma IV is proved by "chasing ratios much in the fashion Desargues was later to use. In this case collinearity could have been established by appealing to the converse of Menelaus' theorem, but when Pappus reached that point he missed that trick and continued to chase ratios until the conclusion was established—in effect, proving the converse of Menelaus' theorem without saying so" [12, pp. 10–1]. I would add that the similarity of "fashion" in Pappus and Desargues is probably due to the latter's having studied the former. Moreover, Pappus seems *not* to have "missed the trick," since he asserts the desired collinearity at a point when it can be recognized only by somebody who knows Menelaus's Theorem.

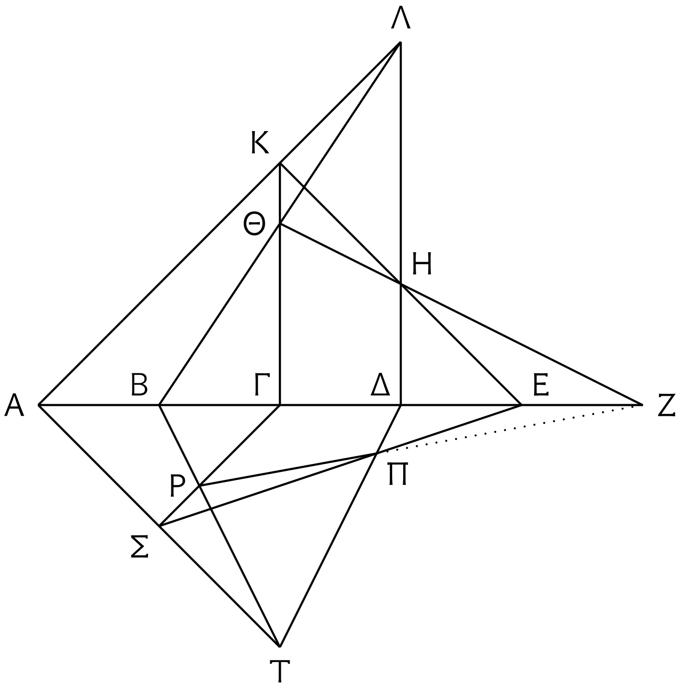


Figure 28. The Quadrangle Theorem

By Thales's Theorem again, the points Θ, Ξ, and Z must be collinear, and therefore the same is true for Θ, H, and Z. This completes the proof of Lemma VIII.

The steps of the proof are reversible. Thus, if we are given the complete quadrangle HΘKΛ of Figure 26 and the points A, B, Γ, Δ, E, and Z where its sides cross a given straight line, the proportion (13) must be satisfied. Therefore if five sides of another complete quadrangle, as ΠΡΣΤ in Figure 28, should pass through the points A, B, Γ, Δ, and E, then the sixth side must pass through Z. This is the **Quadrangle Theorem.**

Wednesday, September 14, 2016

Euclid's proof of Thales's Theorem

Suppose in Figure 29, $DE \parallel BC$. Then $\triangle DEB = \triangle DEC$, so

$$\begin{aligned} AD : DB &:: \triangle ADE : \triangle DBE \\ &:: \triangle ADE : \triangle ECD :: AE : EC. \end{aligned}$$

Conversely, if $AD : DB :: AE : EC$, then

$$\begin{aligned} \triangle ADE : \triangle DBC &:: AD : DB \\ &:: AE : EC :: \triangle ADE : \triangle BCE, \end{aligned}$$

and so $\triangle BCD = \triangle BCE$, which gives us $DE \parallel BC$. This is Euclid's proof of Thales's Theorem.

In any proportion, any of the four terms, be it a straight line, a plane figure, or a solid, can be replaced by a term of

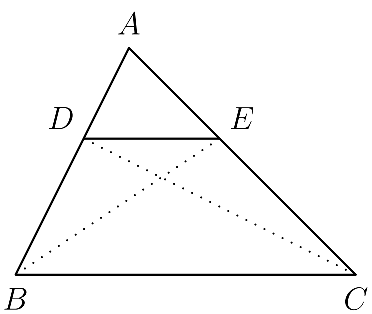


Figure 29. Thales's Theorem

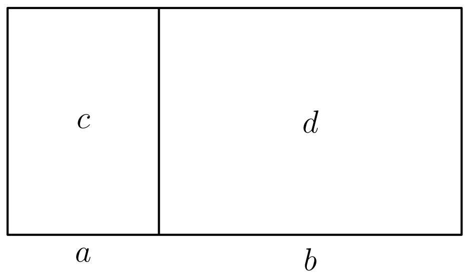


Figure 30. A proportion of lengths and areas

the same kind. Beyond this, the foregoing proof relies on two properties of proportions:

1. If a and b are bounded straight lines, and c and d are plane figures, then the proportion

$$a : b :: c : d$$

is true if and only if there is a rectangle as in Figure 30.

2. Sameness of ratio is indeed an equivalence relation. In particular, it is transitive.

These properties follow from Euclid's definition of proportion, found in Book V of the *Elements.* But this definition relies on the **Archimedean Axiom,** that for any two magnitudes that have a ratio, some multiple of either of them must exceed the other.

A simple definition of proportion

We can avoid using the Archimedean Axiom by, first of all, letting Property 1 above be a definition. Moreover, if a, b, c, and d are all straight lines, we define

$$a : b :: c : d$$

if, for some plane figures e and f,

$$a : b :: e : f, \qquad c : d :: e : f.$$

But we must check the following. Suppose this condition is met, and also g and h are plane figures such that

$$a : b :: g : h.$$

Does it follow that $c : d :: g : h$? It does, as is shown in Figure 31a, which can be completed as in Figure 31b. Here the line across the big parallelogram (which may not be a rectangle) really is straight, so that the the smaller parallelograms are equal in pairs, as shown.

Cross ratio in Pappus

In **Lemma III,** the straight lines $\mathsf{\Theta E}$ and $\mathsf{\Theta\Delta}$ cut the straight lines AB, $\mathsf{\Gamma A}$, and $\mathsf{\Delta A}$ as in Figure 32. The diagram is completed by making

$$\mathsf{K\Lambda} \parallel \mathsf{Z\Gamma A}, \qquad \mathsf{\Lambda M} \parallel \mathsf{\Delta A}.$$

We first have

$$\begin{aligned} \mathsf{EZ} : \mathsf{ZA} &:: \mathsf{E\Theta} : \mathsf{\Theta\Lambda}, \\ \mathsf{AZ} : \mathsf{ZH} &:: \mathsf{\Theta\Lambda} : \mathsf{\Theta M}; \end{aligned} \tag{16}$$

so *ex aequali,*

$$\begin{aligned} \mathsf{EZ} : \mathsf{ZH} &:: \mathsf{E\Theta} : \mathsf{\Theta M}, \\ \mathsf{\Theta E} \cdot \mathsf{HZ} &= \mathsf{EZ} \cdot \mathsf{\Theta M}. \end{aligned}$$

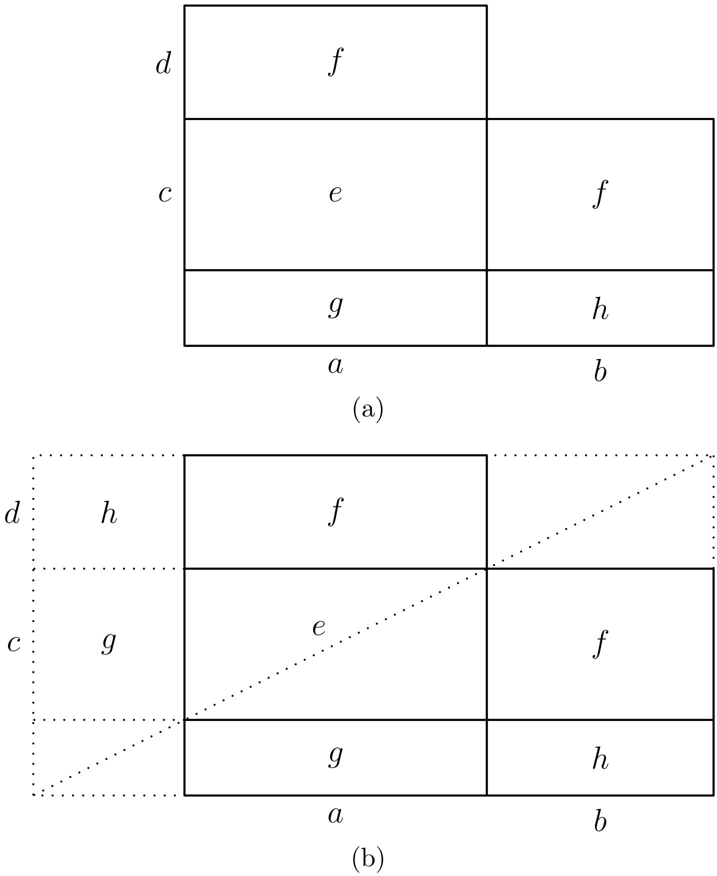


Figure 31. A definition of proportion

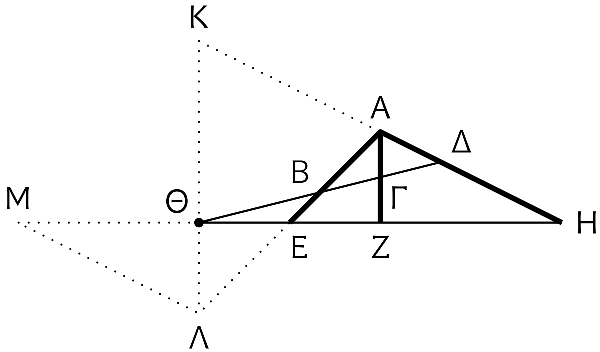


Figure 32. Lemma III

Being equal, these areas have the same ratio to $\mathsf{EZ} \cdot \mathsf{\Theta H}$, and so

$$\begin{aligned} \mathsf{\Theta E} \cdot \mathsf{HZ} : \mathsf{EZ} \cdot \mathsf{\Theta H} &:: \mathsf{EZ} \cdot \mathsf{\Theta M} : \mathsf{EZ} \cdot \mathsf{\Theta H} \\ &:: \mathsf{\Theta M} : \mathsf{\Theta H} \\ &:: \mathsf{\Lambda\Theta} : \mathsf{\Theta K}. \end{aligned} \tag{17}$$

The ratio $\mathsf{\Lambda\Theta} : \mathsf{\Theta K}$ is independent of the choice of the straight line through $\mathsf{\Theta}$ that cuts the three straight lines that pass through A. In particular, we can conclude immediately

$$\mathsf{\Theta E} \cdot \mathsf{HZ} : \mathsf{EZ} \cdot \mathsf{\Theta H} :: \mathsf{\Theta B} \cdot \mathsf{\Delta\Gamma} : \mathsf{B\Gamma} \cdot \mathsf{\Theta\Delta},$$

which is the theorem.

The ratio $\mathsf{\Theta E} \cdot \mathsf{HZ} : \mathsf{EZ} \cdot \mathsf{\Theta H}$ can be called the **cross ratio** (*çapraz oran*) of the points $\mathsf{\Theta}$, E, Z, and H. By the lemma, if four straight lines in a plane intersect at a point, then the cross ratio of the four points where some other straight line crosses the lines is always the same. We can see this using Figure 33.

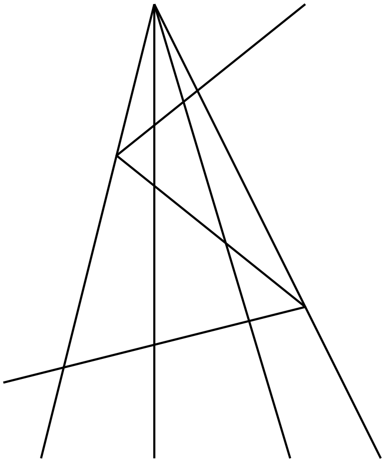


Figure 33. Invariance of cross ratio in Pappus

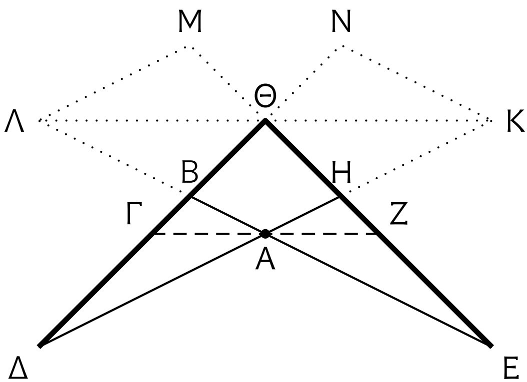


Figure 34. Lemma X

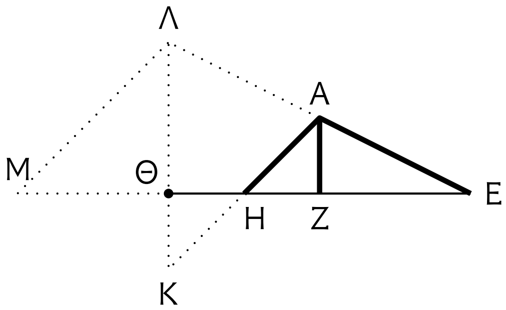


Figure 35. Lemma III reconfigured

Lemma X is a converse to Lemma III. The diagram is as in Figure 34, where the hypothesis is

$$\mathsf{\Theta\Delta \cdot B\Gamma : \Delta\Gamma \cdot B\Theta :: \Theta H \cdot ZE : \Theta E \cdot ZH}. \tag{18}$$

We make $\mathsf{K\Lambda}$ parallel to $\mathsf{\Gamma A}$, and then extend AB and $\mathsf{A\Delta}$ to meet $\mathsf{K\Lambda}$ at two points, which Pappus writes as K and $\mathsf{\Lambda}$, though today we might say this is the wrong order. We ensure $\mathsf{\Lambda M} \parallel \mathsf{A\Delta}$ and $\mathsf{KN} \parallel \mathsf{AB}$, with $\mathsf{E\Theta}$ extended to M, and $\mathsf{\Delta\Theta}$ to N. Now we repeat part of the proof of in Lemma III, at least in the configuration of Figure 35, with two of the three concurrent straight lines interchanged. That is, using only the part of the diagram shown in Figure 36a, we have

$$\begin{aligned}
\mathsf{\Delta\Theta : \Theta N} &:: \mathsf{\Delta\Gamma : \Gamma B},\\
\mathsf{\Delta\Theta \cdot \Gamma B} &= \mathsf{\Delta\Gamma \cdot \Theta N},\\
\mathsf{\Delta\Theta \cdot B\Gamma : \Delta\Gamma \cdot B\Theta} &:: \mathsf{\Gamma\Delta \cdot \Theta N : \Delta\Gamma \cdot B\Theta}\\
&:: \mathsf{\Theta N : \Theta B}\\
&:: \mathsf{K\Theta : \Theta\Lambda}.
\end{aligned}$$

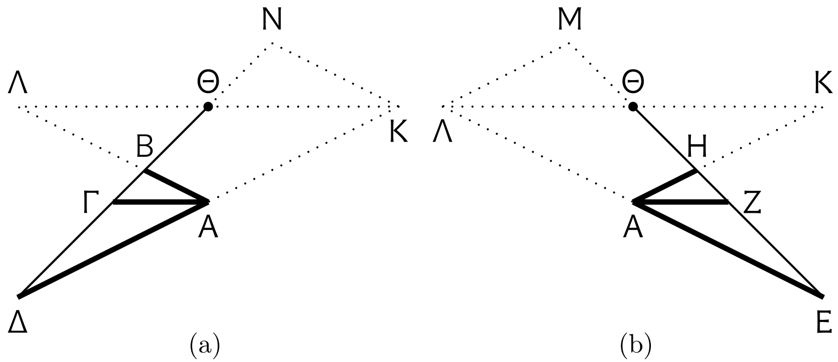


Figure 36. Lemma X: two halves of the proof

Now we move to the part of the diagram shown in Figure 36b, where we have proportions corresponding to the last two:

$$\begin{aligned} \mathsf{K\Theta : \Theta\Lambda} &:: \mathsf{H\Theta : \Theta M} \\ &:: \mathsf{\Theta H \cdot ZE : \Theta M \cdot ZE}. \end{aligned}$$

In sum, we have shown

$$\mathsf{\Delta\Theta \cdot B\Gamma : \Delta\Gamma \cdot B\Theta :: \Theta H \cdot ZE : \Theta M \cdot ZE}.$$

But the left member already appears in (18). Hence the right members of the two proportions are the same, that is,

$$\begin{gathered} \mathsf{\Theta H \cdot ZE : \Theta E \cdot ZH :: \Theta H \cdot ZE : \Theta M \cdot ZE}, \\ \mathsf{\Theta E \cdot ZH = \Theta M \cdot ZE}, \\ \mathsf{\Theta M : \Theta E :: HZ : ZE}. \end{gathered}$$

Thus what we did in Figure 36a, we have done in reverse in Figure 36b. It remains to draw the conclusion $\mathsf{AZ \parallel K\Lambda}$, corresponding to the hypothesis $\mathsf{A\Gamma \parallel K\Lambda}$. Pappus argues as follows (the bracketed proportions replaced with a reference to addition and alternation):

$$\begin{bmatrix} \mathsf{\Theta M} + \mathsf{\Theta E} : \mathsf{\Theta E} :: \mathsf{HZ} + \mathsf{ZE} : \mathsf{ZE}, \\ \mathsf{ME} : \mathsf{\Theta E} :: \mathsf{HE} : \mathsf{ZE}, \end{bmatrix}$$

$$\mathsf{ME} : \mathsf{EH} :: \mathsf{\Theta E} : \mathsf{EZ},$$

$$\mathsf{\Lambda E} : \mathsf{EA} :: \mathsf{\Theta E} : \mathsf{EZ},$$

and so $\mathsf{AZ} \parallel \mathsf{K\Lambda}$, which means $\mathsf{\Gamma AZ}$ must be straight.

Friday, September 16, 2016

Addition and multiplication

In Euclid, a **bounded straight line** (*sınırlanmış doğru çizgi*) is called more simply a **straight line** (*doğru çizgi*), and more simply still, a **“straight”** (*doğru*). In English, this is usually called a **line segment** (*doğru parçası*), although for Euclid, and sometimes in English too, a **line** (*çizgi*) may be curved. For example, a circle is a certain kind of line. For us though, henceforth lines will always be straight and unbounded.

It is clear how to add the lengths of two line segments: place the segments end to end in one straight line. Thus in Figure 37,

$$AB + BC = AC.$$

It is implicit in Euclid and Pappus, and we are making it explicit now, that AB, for example, can be reversed and placed on itself so that A is on B and B is on A: briefly, $AB = BA$. Then

$$BC + AB = CB + BA = CA = AC,$$

so addition is commutative.

We can consider addition in another way. We select a point A on a Euclidean unbounded straight line. If B and C are

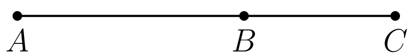


Figure 37. Addition of line segments

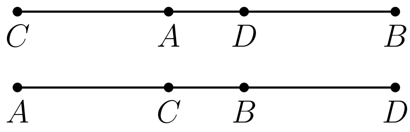


Figure 38. Sum of two points with respect to a third

also points on the line, distinct from one another and from A, then there are two points D on the line such that

$$BD = AC,$$

but for only one of them do we also have

$$CD = AB.$$

See Figure 38. We may refer to this D as the **sum of B and C with respect to A,** writing

$$D = B +_A C = C +_A B.$$

If B and C are the same point, then $B +_A C$ is the unique point D different from A such that $AB = BD$. Finally, if one of B and C is the same as A, then $B +_A C$ is the same as the other. Thus defined, the operation $+_A$ makes the points of the line into an abelian group with neutral element A.

Can we define a multiplication that makes this group into a field? We have already treated the product of two line segments as a rectangle having sides equal to those two segments. But now we should like the product of line segments to be another segment.

If a and b are points on the real number line as studied in calculus, then $a + b$ and $a \cdot b$ are also points on that line. The *directed* segment from a to $a + b$ is equal to that from $\mathsf{0}$ to b. But how is $a \cdot b$ defined?

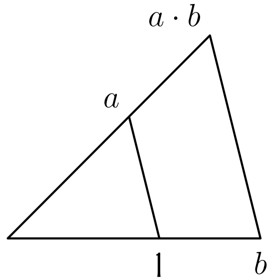


Figure 39. Descartes's definition of multiplication

Descartes showed the way [7]. The position of 1 on the real line must be known. More generally, on our Euclidean straight line with point A chosen, let us select another point B. We can then denote by 1 the length of AB. If a and b are two lengths, then by the proportion

$$1 : a :: b : a \cdot b,$$

we can define the length $a \cdot b$. To be more precise, since this length depends on the segment AB, we might write the length as

$$a \cdot_{AB} b.$$

We can construct a line segment having this length by means of Thales's Theorem, as in Figure 39.

Why is the multiplication $\cdot_{AB}$ commutative? We do know from Thales's Theorem that if A, B, C, and D are now arbitrary line segments, and $A : B :: C : D$, then $A \cdot D = B \cdot C$, where the products are rectangles as before. Since then $B \cdot C = C \cdot B$, we obtain $A : C :: B : D$. Therefore

$$1 : b :: a : a \cdot b,$$

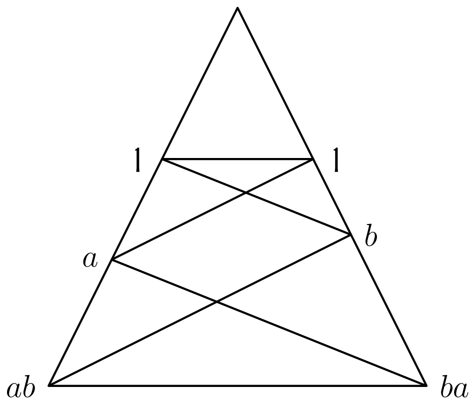


Figure 40. Commutativity of Descartes's multiplication

and so, by interchanging a and b,

$$1 : a :: b : b \cdot a,$$
$$b \cdot a = a \cdot b.$$

For an alternative proof of commutativity of multiplication, we can apply the case of Pappus's Theorem established in Lemma VIII, as in Figure 40.

We can do Euclidean geometry in the set $\mathbb{R}^2$ of ordered pairs of real numbers. We can do some geometry in $\mathbb{K}^2$ for any field $\mathbb{K}$. We can even allow $\mathbb{K}$ to be a non-commutative field (also known as a division ring); but then Pappus's Theorem will not be true in this geometry. We shall see this in more detail tomorrow.

A non-commutative field

Meanwhile, an example of a non-commutative field is $\mathbb{H}$, the field, or rather skew field, of **quaternions,** discovered by

Hamilton. We can obtain this field from the field $\mathbb{C}$ of complex numbers roughly as $\mathbb{C}$ is obtained from $\mathbb{R}$. Indeed, by one definition,

$$\mathbb{C} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x \in \mathbb{R} \ \ \& \ \ y \in \mathbb{R} \right\}.$$

We can write x for $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ and $\mathbf{i}$ for $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x + y\mathbf{i}.$$

Now $\mathbb{C}$ is a commutative field, and it has the automorphism $z \mapsto \overline{z}$ given by

$$\overline{x + y\mathbf{i}} = x - y\mathbf{i}.$$

We define

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z \in \mathbb{C} \ \ \& \ \ w \in \mathbb{C} \right\}.$$

One shows that this is an additive abelian group and is closed under multiplication, and nonzero elements have multiplicative inverses: it is a division ring. We can write z for $\left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix}\right)$ and $\mathbf{j}$ for $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so

$$\begin{pmatrix} z & w \\ -w & z \end{pmatrix} = z + w\mathbf{j}.$$

Then

$$\begin{aligned} \mathbf{ij} &= \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \\ \mathbf{ji} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}. \end{aligned}$$

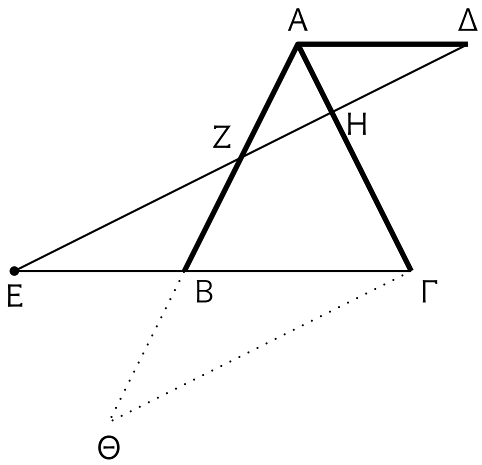


Figure 41. Lemma XI

Intersecting case of Pappus's Theorem

Pappus's **Lemma XI** is that, in Figure 41,

$$\mathsf{\Delta E \cdot ZH : EZ \cdot H\Delta :: \Gamma B : BE.}$$

Since $\mathsf{\Gamma\Theta}$ is drawn parallel to AE, we have the two proportions

$$\mathsf{\Gamma A : AH :: \Gamma\Theta : ZH,}$$
$$\mathsf{\Gamma A : AH :: E\Delta : \Delta H,}$$

and therefore

$$\begin{aligned}\mathsf{E\Delta : \Delta H} &:: \mathsf{\Theta\Gamma : ZH,}\\ \mathsf{E\Delta \cdot ZH} &= \mathsf{\Gamma\Theta \cdot \Delta H,}\\ \mathsf{E\Delta \cdot ZH : \Delta H \cdot EZ} &:: \mathsf{\Gamma\Theta \cdot \Delta H : \Delta H \cdot EZ}\\ &:: \mathsf{\Gamma\Theta : EZ}\\ &:: \mathsf{\Gamma B : BE,}\end{aligned}$$

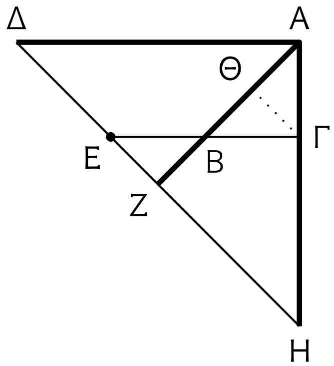


Figure 42. Lemma XI variant

as desired. By Lemma III, if ΕΓ meets ΑΔ at a point Κ, then

$$\mathsf{E\Delta \cdot ZH : EZ \cdot H\Delta :: EK \cdot \Gamma B : EB \cdot \Gamma K}$$
$$\mathsf{:: \Gamma B : EB \;\;\&\;\; EK : \Gamma K.}$$

In Lemma XI, the point K has been sent off to infinity, which means ΕΚ : ΓΚ has become the unit ratio.

Pappus alludes to another case, presumably as in Figure 42. There is no change in the proof.

Lemma XII is that, in Figure 43, the points Η, Μ, and Κ are on a straight line. The proof considers the parts of the diagram shown in Figure 44. Pappus's argument is:

a) By Lemma XI,

$$\mathsf{\Delta Z : Z\Gamma :: \Gamma E \cdot H\Theta : \Gamma H \cdot \Theta E.} \tag{19}$$

b) By Lemma XI again, inversion, and (19),

$$\mathsf{\Gamma Z : Z\Delta :: \Delta E \cdot \Lambda K : \Delta K \cdot \Lambda E,}$$
$$\mathsf{\Delta Z : Z\Gamma :: \Delta K \cdot \Lambda E : \Delta E \cdot \Lambda K,}$$
$$\mathsf{\Gamma E \cdot H\Theta : \Gamma H \cdot \Theta E :: \Delta K \cdot \Lambda E : \Delta E \cdot K\Lambda.}$$

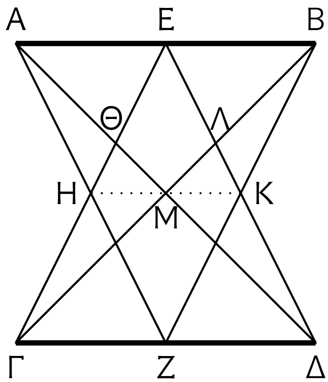


Figure 43. Lemma XII

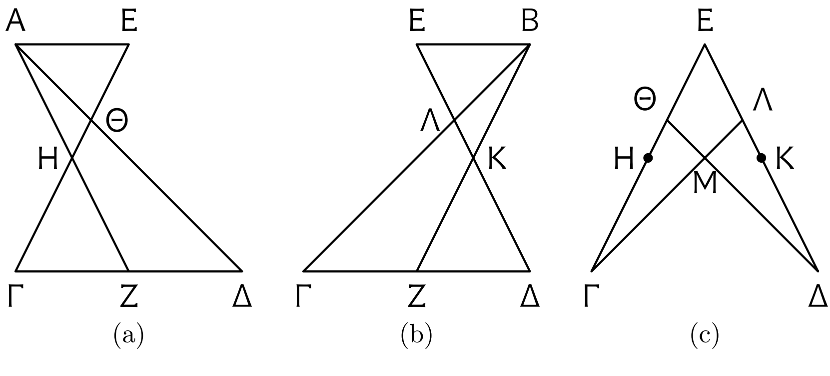


Figure 44. Steps of Lemma XII

c) By Lemma X then, HMK is straight.

Lemma XIII is similar, with Lemma III used in place of Lemma XI. This gives us Pappus's Theorem as proved by Pappus, in the three cases labelled in Figure 24 on page 46.

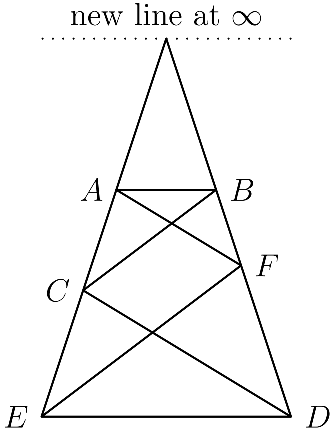
A second proof of Pappus's Theorem

Alternatively, from Lemma VIII, except for the easier case where everything that might be parallel is parallel, all other cases of Pappus's Theorem can be derived by *projection.*

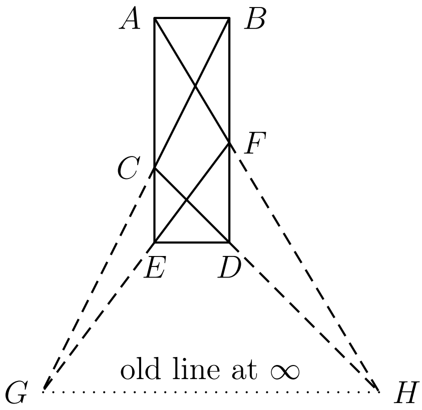
If a diagram is drawn on a transparent notebook cover, and the cover is raised at an angle to the first page, and a shadow of the diagram is cast on that page, all straight lines will remain straight, but some parallel lines will cease to be so, and some intersecting lines will become parallel.

Specifically, we can choose one straight line in a diagram that will become the line at infinity for a new diagram. Call this line in the original diagram A. All straight lines parallel to A will remain parallel in the new diagram. Also, one line parallel to these in the new diagram will represent the line at infinity of the old diagram. Call this line in the new diagram B. Lines that intersected on A become parallel in the new diagram; lines that were parallel to one another, but not to A, will intersect on B. See for example Figure 45.

Tomorrow we shall obtain a third proof of Pappus's Theorem by coordinatizing the projective plane.



(a) Lemma VIII



(b) A new case

Figure 45. Pappus's Theorem by projection

Saturday, September 17, 2016

Cartesian coordinates

We saw yesterday that by choosing a point A on an infinite straight line, we could make the points of the line into an abelian group with the addition denoted by $+_A$. By choosing another point B on the line, we obtained a multiplication $\cdot_{AB}$ of lengths. This is not quite a multiplication of points. However, given points C and D on the line, we can define

$$C \cdot_{AB} D = E,$$

where E is a point on the line such that, if a and b are the lengths of AC and AD, then $a \cdot_{AB} b$ is the length of AE, and also A lies between B and E just in case it lies between C and D. Equipped with addition and multiplication so defined, the Euclidean line becomes a field.

We can obtain an isomorphic field by considering ratios of line segments. On Wednesday we defined ratios of line segments and of rectangles. By this definition, any two line segments have the ratio of two rectangles. Conversely, for any two rectangles, we can find line segments that have their ratio. For example, in Figure 46,

$$ABFE : DFKH :: AB : BC$$

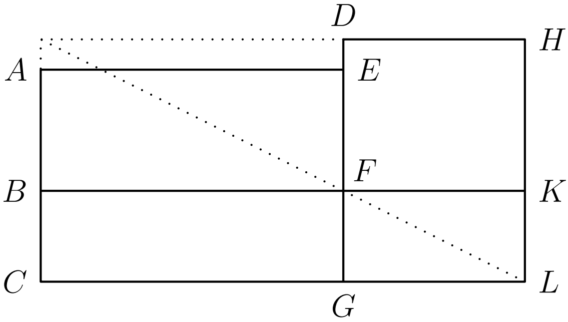


Figure 46. Line segments in the ratio of rectangles

since $DFKH = BCGF$. Ratios constitute a group, the operation being composition, since

$$(a : b \ \& \ b : c) \ \& \ c : d :: a : c \ \& \ c : d :: a : d,$$
$$a : b \ \& \ (b : c \ \& \ c : d) :: a : b \ \& \ b : d :: a : d.$$

Composition is commutative, since if

$$a : b :: c : d,$$

then by Thales's Theorem

$$ad = bc, \qquad a : c :: b : d$$

(see Figure 47), so that

$$a : b \ \& \ b : c :: a : c,$$
$$b : c \ \& \ a : b :: b : c \ \& \ c : d :: b : d :: a : c.$$

We have defined the product of two line segments in two different ways:

1) as the rectangle bounded by them, or

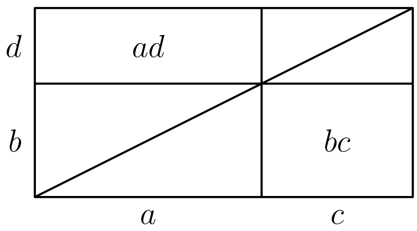


Figure 47. Alternation of ratio

2) as the straight line that has the same ratio to one of the two segments that the other one has to some predetermined segment.

In the former case, commutativity is immediate; in the latter case, it follows from the theorems of Thales and Pappus, as we showed. In either case, we can define sums of ratios of line segments, through the usual process of finding a common denominator:

$$(a : b) + (c : d) = ad + bc : bd.$$

We get the same result, no matter which multiplication of segments we use.

If we allow negative lengths and a zero length; then the ratios of straight lines compose a field, which we shall call $\mathbb{K}$.

We shall now work in the **Euclidean plane:** the plane in which the propositions of Book I of the *Elements* are true. If we choose a neutral point A, we obtain an additive abelian group, as we did on a line. Associativity can be checked as in Figure 48. The group can also be understood as a vector space over the field $\mathbb{K}$ of ratios. Here if s is such an ratio, and B is a point of the plane different from A, we define

$$s \cdot_A B = C,$$

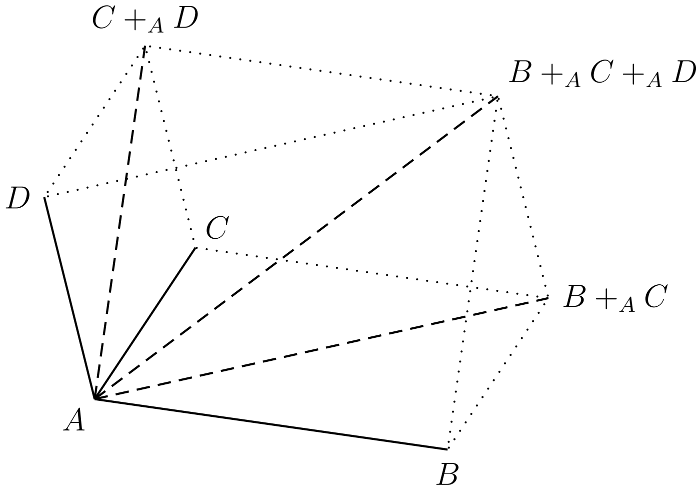


Figure 48. Associativity of addition

where C is the point on AB such that

$$AC : AB = s.$$

Now suppose A, B, and C are vertices of a triangle, that is, none of the three is on the straight line determined by the other two. Then B and C compose a basis of the vector space whose neutral point is A. In particular, for any point D of the plane, there is a unique ordered pair (s, t) of ratios, meaning $(s, t) \in \mathbb{K}^2$, such that

$$D = s \cdot_A B +_A t \cdot_A C.$$

Here (s, t) is the pair of **Cartesian coordinates** of D with respect to ABC. Conversely, every element of $\mathbb{K}^2$ corresponds to a point of the plane in this way.

Barycentric coordinates

Again, we have chosen triangle ABC, and for some D in the same plane we have

$$D = s \cdot_A B +_A t \cdot_A C.$$

This fails if we replace A with some other point. However, the equations

$$\begin{aligned} D &= A +_A s \cdot_A (B -_A A) +_A t \cdot_A (C -_A A) \\ &= (1 - s - t) \cdot_A A +_A s \cdot_A B +_A t \cdot_A C \end{aligned}$$

are still correct if the subscripts A are replaced with any other point. Thus we may write simply

$$D = (1 - s - t)A + sB + tC.$$

Conversely, if p, q, and r are three ratios whose sum is 1, then the linear combination

$$pA + qB + rC$$

is unambiguous. Considering ABC as fixed, we may write the same point $pA + qB + rC$ as

$$(p : q : r).$$

But now we can allow

$$(p : q : r) = (tp : tq : tr)$$

if $t \neq 0$. Thus, given arbitrary ratios p, q, and r whose sum is not 0, we have

$$\begin{aligned} (p : q : r) &= \frac{p}{p+q+r}A + \frac{q}{p+q+r}B + \frac{r}{p+q+r}C \\ &= \frac{q}{p+q+r} \cdot_A B +_A \frac{r}{p+q+r} \cdot_A C. \end{aligned}$$

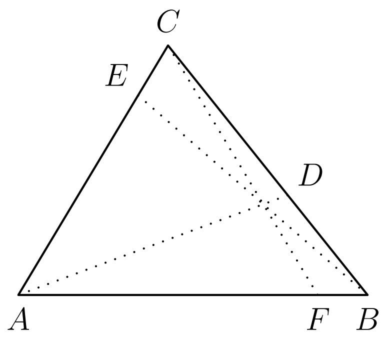


Figure 49. Ceva's Theorem

This point has the **barycentric coordinates** p, q, and r, but each must be considered together with the sum $p+q+r$. The idea is that the point is the center of gravity (the barycenter) of the system with weights p, q, and r at A, B, and C respectively. If we define

$$D = \frac{q}{q+r}B + \frac{r}{q+r}C,$$

then D is a point on BC, and

$$(p:q:r) = \frac{p}{p+q+r}A + \frac{q+r}{p+q+r}D,$$

so $(p:q:r)$ is a point on AD. Similarly we can define E and F on AC and AB respectively, so that $(p:q:r)$ is on BE and CF. Note that

$$BD : DC :: r : q,$$

and so on, so that we have **Ceva's Theorem:** in Figure 49, the lines AD, BE, and CF have a common point if and only if

$$BD : DC \;\;\&\;\; CE : EA \;\;\&\;\; AF : FB :: 1.$$

We can understand $(p : q : r)$ as the equivalence class consisting of all ordered triples (px, qx, rx), where x is a nonzero ratio. The notation makes sense for all ratios p, q, and r. We are going to show how to extend the Euclidean plane to the projective plane by giving geometric meaning to $(p : q : r)$ when $p + q + r = 0$ but at least one of p, q, and r is not 0.

Projective coordinates

For every straight line in the plane, there are ratios a, b, and c, where at least one of a and b is not 0, such that the straight line consists of the points such that, if their Cartesian coordinates are (s, t), then

$$as + bt + c = 0.$$

If the same point has barycentric coordinates $(p : q : r)$, where $p + q + r = 1$, then $(s, t) = (q, r)$, and so

$$\begin{aligned} 0 &= aq + br + c(p + q + r) \\ &= cp + (a + c)q + (b + c)r. \end{aligned}$$

Thus the same line is given by

$$ax + by + c = 0$$

in Cartesian coordinates and

$$cx + (a + c)y + (b + c)z = 0$$

in barycentric coordinates. The straight lines parallel to this one are obtained by changing c alone. Relabelling, we now have that every straight line is given by an equation

$$ax + by + cz = 0$$

in barycentric coordinates, where one of the coefficients a, b, and c is different from the others (the coefficients are not all equal). We obtain parallel lines by adding the same ratio to each coefficient.

Thus, if $ap + bq + cr = 0$, and $p + q + r = 0$, but $(p, q, r) \neq (0, 0, 0)$, then $(p : q : r)$ satisfies the equation of every straight line parallel to $ax + by + cz = 0$, and no other straight line. We can understand $(p : q : r)$ as the point at infinity of the straight lines parallel to $ax + by + cz = 0$. The line at infinity is then defined by $x + y + z = 0$.

The projective plane now consists of points $(p : q : r)$, where $(p, q, r) \neq (0, 0, 0)$. The expression $(p : q : r)$ consists of **projective coordinates** for the point.

There is now a one-to-one correspondence between points of the projective plane and straight lines in that plane: if the point is $(p : q : r)$ in projective coordinates, the straight line is given by $px + qy + rz = 0$.

We may use the notation

$$\mathbb{P}^2(\mathbb{K}) = \Big\{(x : y : z)\colon (x, y, z) \in \mathbb{K}^3 \setminus \big\{(0, 0, 0)\big\}\Big\}.$$

Any triangle ABC in the Euclidean plane determines a system of barycentric coordinates for the points of the plane, hence a system of projective coordinates for the projective plane. However, suppose a fourth point D in the Euclidean plane does not lie on any of the three sides of ABC. Then D has barycentric coordinates $(\mu : \nu : \rho)$, with $\mu\nu\rho \neq 0$. There is now a bijection

$$(x : y : z) \mapsto \left(\frac{x}{\mu} : \frac{y}{\nu} : \frac{z}{\rho}\right)$$

from the set of points of the projective plane to itself. This bijection takes the line $ax + by + cz = 0$ to the line $\mu ax +$

$\nu by + \rho cz = 0$, while fixing A, B, and C, which are $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ respectively. The points that used to be on the line at infinity are sent to the line

$$\mu x + \nu y + \rho z = 0,$$

while of course the points now at infinity are on $x+y+z=0$. The only former point at infinity that is still at infinity is $(\nu - \rho : \rho - \mu : \mu - \nu)$. In particular, if E is a point that used to be at infinity, we can have chosen D so that E is no longer at infinity.

In this way, for any quadrangle $ABCD$ in the projective plane, there is a system of projective coordinates in which the vertices of the quadrangle are $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, and $(1:1:1)$ respectively.

A third proof of Pappus's Theorem

To prove Pappus's Hexagon Theorem in yet another way, again we suppose the vertices of the hexagon $ABCDEF$ lie alternately on straight lines ACE and BDF, and we let

- AB and DE intersect at G,
- BC and EF intersect at H,
- CD and FA intersect at K.

We may now suppose further

$$\begin{aligned} A &= (1:1:1), & B &= (1:0:0), \\ E &= (0:1:0), & K &= (0:0:1), \end{aligned}$$

as in Figure 50. The point of the choice is that no line of the diagram contains two of B, E, and K, but these are on the lines through A. If B, E, and K are all on a straight line, we

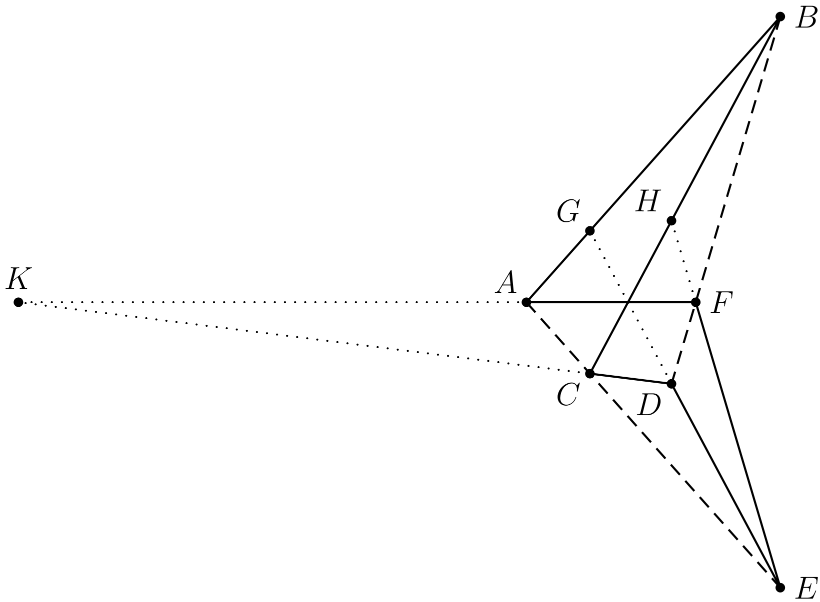


Figure 50. Pappus's Hexagon Theorem

may replace them with F, C, and H. For the lines through A we have

$$AB = \{y = z\}, \qquad AE = \{x = z\}, \qquad AK = \{x = y\},$$

and so, for some p, q, and r,

$$G = (p : 1 : 1), \qquad C = (1 : q : 1), \qquad F = (1 : 1 : r).$$

Consequently,

$$BF = \{z = ry\}, \quad EG = \{x = pz\}, \quad KC = \{y = qx\}.$$

Since these three lines have a common point, namely D, we must have $D = (1 : q : rq) = (pr : 1 : r) = (p : qp : 1)$, and so

$$1 = prq = qpr = rqp.$$

But we have also

$$BC = \{y = qz\}, \quad EF = \{z = rx\}, \quad KG = \{x = py\}.$$

In particular, H being the intersection of the first two of these lines, we have

$$H = (1 : qr : r),$$

and this lies on KG since $pqr = 1$. Figure 50 shows the situation when $p = 2$ and $q = 2$, so $r = 1/4$.

Sunday, September 18, 2016

The projective plane over a field

Given a field $\mathbb{K}$ (possibly a "skew" or non-commutative field), we have
$$\mathbb{P}^2(\mathbb{K}) = \Big(\mathbb{K}^3 \setminus \{(0,0,0)\}\Big)/{\sim},$$
where
$$(p,q,r) \sim (p',q',r')$$
if and only if, for some t in $\mathbb{K} \setminus \{0\}$,
$$p' = tp, \qquad q' = tq, \qquad r' = tr.$$

We may understand this both as the set of points in the projective plane and as the set of lines in the projective plane. The point $(p : q : r)$ sits on the line $(a : b : c)$ if and only if $ap + bq + cr = 0$. The line at infinity is $(1 : 1 : 1)$, that is, the line given by the equation $x + y + z = 0$. So the points at infinity satisfy this.

The Fano Plane

For the simplest example, we may let $\mathbb{K}$ be the two-element field $\mathbb{F}_2$, thus obtaining the **Fano Plane.** The four points of ${\mathbb{F}_2}^2$ in barycentric coordinates are
$$(1 : 0 : 0), \quad (0 : 1 : 0), \quad (0 : 0 : 1), \quad (1 : 1 : 1).$$

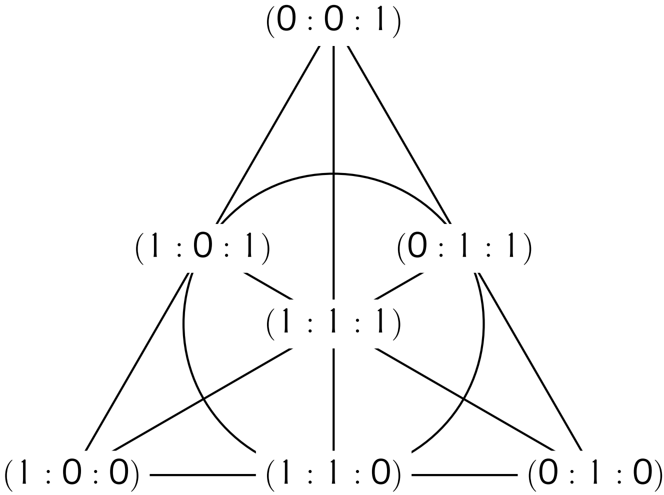


Figure 51. The Fano Plane

There are three points at infinity in $\mathbb{P}^2(\mathbb{K})$:

$$(1:1:0), \qquad (1:0:1), \qquad (0:1:1).$$

If we call these A, B, C, D, E, F, and G respectively, then the seven lines are as follows:

$$\begin{aligned} \{x=0\} &= BCG, & \{y=z\} &= ADG, \\ \{y=0\} &= ACF, & \{x=z\} &= BDF, \\ \{z=0\} &= ABE, & \{x=y\} &= CDE, \end{aligned}$$

and the line at infinity, EFG. All of this can be depicted as in Figure 51.

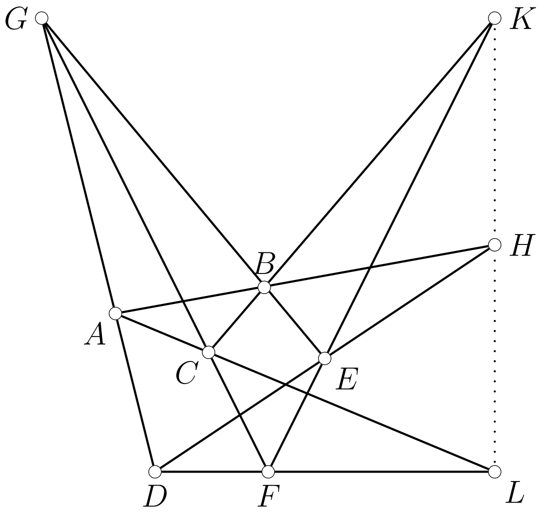


Figure 52. Desargues's Theorem

A proof of Desargues's Theorem

We now prove Desargues's Theorem in $\mathbb{P}^2(\mathbb{K})$ for arbitrary $\mathbb{K}$. The diagram is as in Figure 14, repeated as Figure 52. In $\mathbb{K}^3$, if triangles ABC and DEF lie in two different planes, then each of H, K, and L lies in each of those planes, and therefore the three points lie in the intersection of the two planes, which is a straight line. (If the planes are parallel, they meet at their common line at infinity.) If ABC and DEF lie in the same plane, the diagram can still be considered as the "shadow" or projection of the three-dimensional case. So Desargues's Theorem holds in $\mathbb{P}^2(\mathbb{K})$.

If $\mathbb{K}$ is not commutative, then Pappus's Theorem does not hold in $\mathbb{P}^2(\mathbb{K})$. However, with three applications of Pappus's Theorem alone, we can prove Desargues's Theorem as sketched in Figure 53. using in turn the hexagons $ABGMDC$,

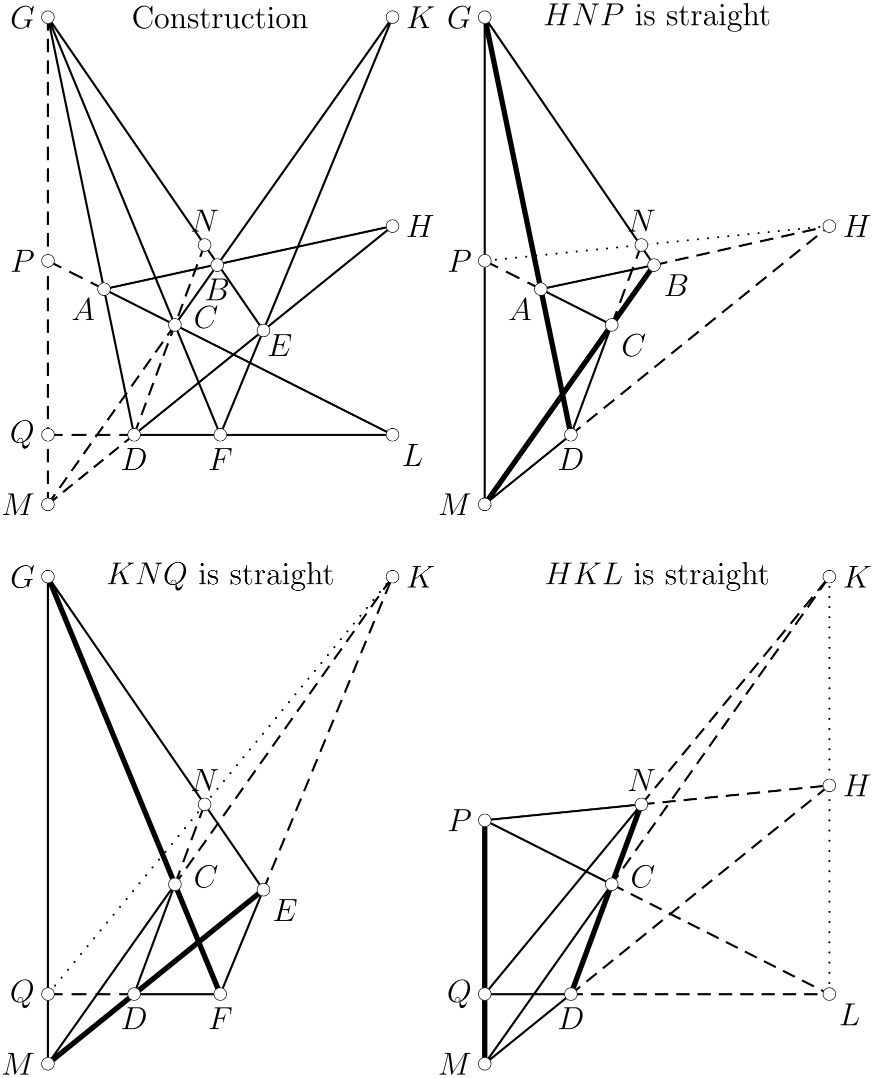


Figure 53. Proof of Desargues's Theorem

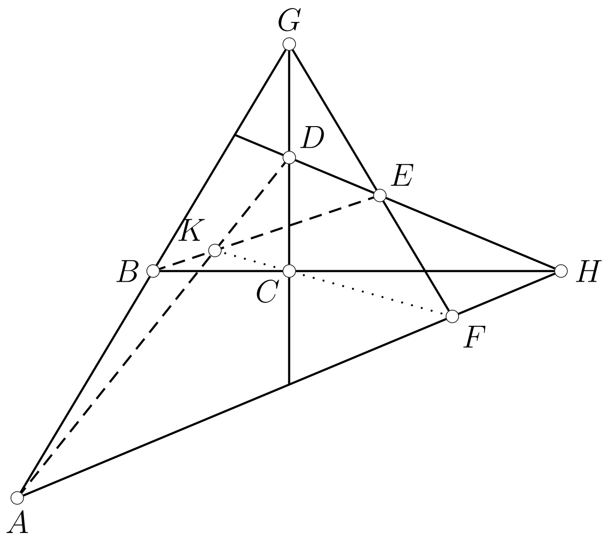


Figure 54. The dual of Pappus's Theorem

$CDFEGM$, and $CMDQNP$. Strictly, we have now proved Desargues's Theorem on the assumption of three axioms:

1. Two points determine a line.
2. Two lines determine a point.
3. Pappus's Theorem is true.

Duality

The **dual** of a statement about the projective plane is obtained by interchanging points and lines. Thus the dual of Pappus's Theorem is that if the sides of a hexagon alternately meet two points, then the straight lines met by pairs of opposite vertices meet a common point. So, in the hexagon $ABCDEF$, let AB, CD, and EF intersect at G, and let BC, DE, and FA intersect at H, as in Figure 54. If the diagonals AD and

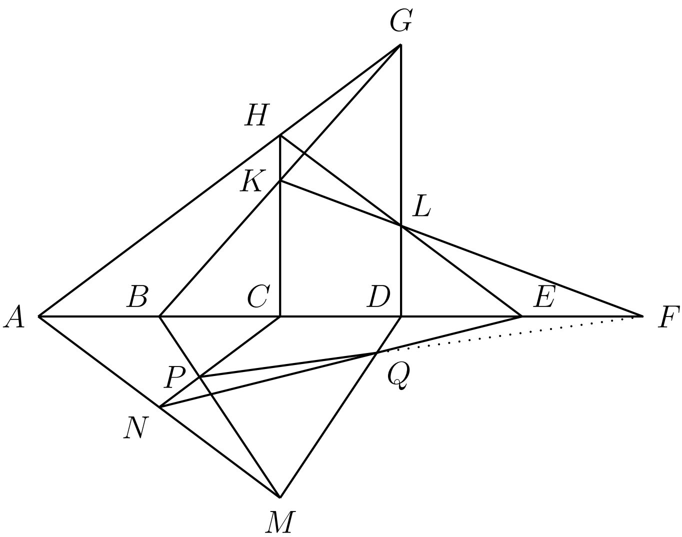


Figure 55. The Quadrangle Theorem

BE meet at K, then the diagonal CF also passes through K. For we can apply Pappus's Theorem itself to the hexagon $ADGEBH$, since AGB and DEH are straight. Since AD and EB intersect at K, and DG and BH at C, and GE and HA at F, it follows that KCF is straight.

It now follows from the three axioms above that the dual of Desargues's Theorem is true. But the dual is precisely the converse.

A second proof of the Quadrangle Theorem

We now use the converse of Desargues's Theorem twice, and the original Theorem once, to prove the Quadrangle Theorem. We shall show that, in Figure 55, the line PQ passes through F.

1. by the converse of Desargues's Theorem applied to triangles GHL and MNQ, since
   - GH and MN meet at A,
   - GL and MQ meet at D, and
   - HL and NQ meet at E,

   and ADE is straight, it follows that GM, HN, and LQ intersect at a common point R (not drawn).
2. Likewise, in triangles GHK and MNP, since
   - GH and MN meet at A,
   - GK and MP meet at B, and
   - HK and NP meet at C,

   and ABC is straight, it follows that KP passes through the intersection point of GM and HN, which is R.
3. So now we know that HN, KP, and LQ intersect at R. Therefore, by Desargues's Theorem, the respective sides of triangles HKL and NPQ intersect along a straight line. But HK and NP intersect at C, and HL and NQ intersect at E; and KL intersects CE at F; therefore PQ must also intersect CE at F.

Bibliography

[1] Apollonius of Perga. *On conic sections.* Great Books of the Western World, no. 11. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

[2] Apollonius of Perga. *Conics. Books I–III.* Green Lion Press, Santa Fe, NM, revised edition, 1998. Translated and with a note and an appendix by R. Catesby Taliaferro, with a preface by Dana Densmore and William H. Donahue, an introduction by Harvey Flaumenhaft, and diagrams by Donahue, edited by Densmore.

[3] M. Chasles. *Les trois livres de porismes d'Euclide.* Mallet-Bachelier, Paris, 1860. Electronic source `gallica.bnf.fr` / Bibliothèque de l'Ecole polytechnique.

[4] Frances Marguerite Clarke and David Eugene Smith. "Essay pour les Coniques" of Blaise Pascal. *Isis*, 10(1):16–20, March 1928. Translation by Clarke, Introductory Note by Smith. `www.jstor.org/stable/224736`.

[5] H. S. M. Coxeter. *Introduction to Geometry.* John Wiley & Sons, New York, second edition, 1969. First edition, 1961.

[6] Arno Cronheim. A proof of Hessenberg's theorem. *Proc. Amer. Math. Soc.*, 4:219–221, 1953.

[7] René Descartes. *The Geometry of René Descartes.* Dover Publications, Inc., New York, 1954. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

[8] René Descartes. *The Philosophical Writings of Descartes*, volume I. Cambridge University Press, 1985. translated by John Cottingham, Robert Stoothoff, and Dugald Murdoch.

[9] René Descartes. *La Géométrie.* Jacques Gabay, Sceaux, France, 1991. Reprint of Hermann edition of 1886.

[10] Diogenes Laertius. *Lives of Eminent Philosophers.* Number 184 in the Loeb Classical Library. Harvard University Press and William Heinemann Ltd., Cambridge, Massachusetts, and London, 1959. Volume I of two. With an English translation by R. D. Hicks. First published 1925. Available from `archive.org` and `www.perseus.tufts.edu`.

[11] Euclid. *The Thirteen Books of Euclid's* Elements. Dover Publications, New York, 1956. Translated from the text of Heiberg with introduction and commentary by Thomas L. Heath. In three volumes. Republication of the second edition of 1925. First edition 1908.

[12] J. V. Field and J. J. Gray. *The geometrical work of Girard Desargues.* Springer-Verlag, New York, 1987.

[13] Thomas Heath. *A History of Greek Mathematics. Vol. II. From Aristarchus to Diophantus.* Dover Publications Inc., New York, 1981. Corrected reprint of the 1921 original.

[14] David Hilbert. *The Foundations of Geometry.* Authorized translation by E. J. Townsend. Reprint edition. The Open Court Publishing Co., La Salle, Ill., 1959. Based on lectures 1898–99. Translation copyrighted 1902. Project Gutenberg edition released December 23, 2005 (`www.gutenberg.net`).

[15] Victor J. Katz, editor. *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook.* Princeton University Press, Princeton and Oxford, 2007.

[16] Morris Kline. *Mathematical Thought from Ancient to Modern Times.* Oxford University Press, New York, 1972.

[17] Reviel Netz. *The Shaping of Deduction in Greek Mathematics*, volume 51 of *Ideas in Context.* Cambridge University Press, Cambridge, 1999. A study in cognitive history.

[18] Pappus of Alexandria. *Pappus Alexandrini Collectionis Quae Supersunt*, volume II. Weidmann, Berlin, 1877. E libris manu scriptis edidit, Latina interpretatione et commentariis instruxit Fridericus Hultsch.

[19] Pappus of Alexandria. *Book 7 of the Collection. Part 1. Introduction, Text, and Translation.* Springer Science+Business Media, New

York, 1986. Edited With Translation and Commentary by Alexander Jones.

[20] Pappus of Alexandria. *Book 7 of the Collection. Part 2. Commentary, Index, and Figures.* Springer Science+Business Media, New York, 1986. Edited With Translation and Commentary by Alexander Jones. Pages numbered continuously with Part 1.

[21] Dimitris Patsopoulos and Tasos Patronis. The theorem of Thales: A study of the naming of theorems in school geometry textbooks. *The International Journal for the History of Mathematics Education*, 1(1), 2006. `www.comap.com/historyjournal/index.html`.

[22] David Pierce. History of mathematics: Log of a course. `mat.msgsu.edu.tr/~dpierce/Courses/Math-history/`, June 2011. Edited September 2013. 204 pp., size A5, 10-pt type.

[23] David Pierce. St John's College. *The De Morgan Journal*, 2(2):62–72, 2012. Available from `education.lms.ac.uk/wp-content/uploads/2012/02/st-johns-college.pdf`, accessed October 1, 2014.

[24] David Pierce. Abscissas and ordinates. *J. Humanist. Math.*, 5(1):223–264, 2015. Available at `scholarship.claremont.edu/jhm/vol5/iss1/14`.

[25] David Pierce. Commensurability and symmetry. `mat.msgsu.edu.tr/~dpierce/Geometry/`, July 2016. Submitted for publication. 61 pp., size A5, 12-pt type.

[26] Proclus. *A Commentary on the First Book of Euclid's* Elements. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1992. Translated from the Greek and with an introduction and notes by Glenn R. Morrow. Reprint of the 1970 edition. With a foreword by Ian Mueller.

[27] Ptolemy. The Almagest. In Robert Maynard Hutchins, editor, *Ptolemy Copernicus Kepler*, volume 16 of *Great Books of the Western World*, pages 1–478. Encyclopædia Britannica, Chicago, 1952. Translated by R. Catesby Taliaferro.

[28] Ptolemy. *Ptolemy's Almagest.* Princeton University Press, 1998. Translated and annotated by G. J. Toomer. With a foreward by

Owen Gingerich. Originally published by Gerald Duckworth, London, 1984.

[29] Rene Taton. L' « Essay pour les Coniques » de Pascal. *Revue d'histoire de science et de leurs applications*, 8(1):1–18, 1955. `www.persee.fr/web/revues/home/prescript/article/rhs_0048-7996_1955_num_8_1_3488`.

[30] Ivor Thomas, editor. *Selections Illustrating the History of Greek Mathematics. Vol. II. From Aristarchus to Pappus.* Number 362 in the Loeb Classical Library. Harvard University Press, Cambridge, Mass, 1951. With an English translation by the editor.

[31] Arthur W. Weeks and Jackson B. Adkins. *A Course in Geometry: Plane and Solid.* Ginn and Company, Lexington, Massachusetts, 1970.