Elements of the theory of models

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0 Preface

These notes are written according to my own understanding and preferences, and they should be considered only as a rough draft. I aim to present some logic as mathematics, and I assume that the reader already has some experience with mathematics. I claim no originality, but I do not happen to know of a published treatment that is quite like mine. However, § 2 in particular is influenced by [2] and [5]; most model-theory texts seem not to deal specifically with propositional logic. For the model-theoretic development of first-order logic, see also the early parts of [4], [7] or [6]. Books on logic itself that I have found useful are [3] and [1].

Technical terms in boldface are being defined, perhaps implicitly. Mathematical propositions (theorems, lemmas) whose proofs are not supplied are to be proved by the reader.

1 Conventions

In these notes, the symbol $\iff$ is just an abbreviation for the words 'if and only if'.

Let us denote the set $\{0, 1, 2, \dots\}$ of natural numbers by ω. It is notationally convenient to consider this as the smallest set of sets that contains $\varnothing$ and is closed under the successor-operation, namely

$$A \mapsto A \cup \{A\}.$$

So ω contains $\varnothing$, $\{\varnothing\}$ and $\{\varnothing,\{\varnothing\}\}$, denoted 0, 1 and 2 respectively. Thus, 2 is $\{0,1\}$, which we shall consider as the underlying set—the *universe*—of the 2-element field, $\mathbb{F}_2$. We can write any n in ω as

$$\{0,\dots,n-1\}.$$

Suppose M is a set, and I is a finite set. We shall denote the set of functions from I to M by M^I. Elements of this are **I-tuples**; a typical I-tuple can be written

$$(a_j : j \in I)$$

or just $\mathbf{a}$. If $I = n$ for some n in ω, then we may write $\mathbf{a}$ as $(a_0,\dots,a_{n-1})$ or as the **string** $a_0 \dots a_{n-1}$. As a special case, we have $M^0 = \{0\} = 1$.

2 Propositional model-theory

We first select a set V, and we shall refer to its members as **variables**. Usually, V is countably infinite, but this will not be required in any definitions. If $A \subseteq V$, then we may refer to the ordered pair (A, V) as a **(propositional) structure (for V)** and denote it by

$$\mathfrak{A}.$$

The structure $\mathfrak{A}$ determines, and is determined by, the characteristic function $\chi_{\mathfrak{A}} : V \to 2$, which is given by

$$\chi_{\mathfrak{A}}(P) = \begin{cases} 0, & \text{if } P \in V \smallsetminus A; \\ 1, & \text{if } P \in A. \end{cases}$$

We may call such a characteristic function a **truth-assignment for V**, reading 0 as 'false', and 1 as 'true'.

Remark 2.1. Instead of $\chi_{\mathfrak{A}}$, one may write χ_A if the domain of the function is clear.

Next, we introduce a set

$$\{\wedge, \vee, \neg, \to, \leftrightarrow\}$$

of **Boolean connectives**, along with a pair $\{(,)\}$ of **brackets**. We assume that the Boolean connectives and the brackets are not variables. We define the **(propositional) formulas (for V)** to be the members of the smallest set Φ of strings of elements of $V \cup \{\wedge, \vee, \neg, \to, \leftrightarrow\} \cup \{(,)\}$ such that

(0) $V \subseteq \Phi$;

(1) if $F \in \Phi$, then $\neg F \in \Phi$;

(2) if $F, G \in \Phi$, and $* \in \{\wedge, \vee, \to, \leftrightarrow\}$, then $(F * G) \in \Phi$.

So, we can call $\neg$ a **unary** connective; the other connectives are **binary**. Let us denote the set of propositional formulas in V by $\mathrm{PF}(V)$. The definition of $\mathrm{PF}(V)$ obviously allows proof by induction. That is, if $\Phi \subseteq \mathrm{PF}(V)$, and if Φ satisfies the three enumerated conditions in the definition of $\mathrm{PF}(V)$, then $\Phi = \mathrm{PF}(V)$.

Remark 2.2. The symbols $\wedge$, $\vee$, $\neg$, $\to$ and $\leftrightarrow$ can be read 'and', 'or', 'not', 'implies' and 'if and only if', respectively. We could avoid brackets by using the so-called **Polish notation**, in which we would write $*FG$ instead of $(F * G)$; using **reverse** Polish notation, we would write $FG*$. Note that the formulation $FG\wedge$ in reverse Polish notation could be read as 'F, G too,' or in Turkish as 'F, G de'.

Lemma 2.3 (unique readability). *Any propositional formula that is not a variable is the application of exactly one connective in exactly one way, that is, if F, F', G, G' and H are formulas, then $F * G$ and $\neg H$ cannot be the same formula, and if $F * G$ and $F' *' G'$ are the same formula, then F and F' are the same formula.*

Lemma 2.4 (definition by recursion). *Functions on $\mathrm{PF}(V)$ can be defined recursively. To be precise, suppose S is a set, and $f_\neg$ is a function from S to itself, and f_* is a function from $S \times S$ to S for each binary Boolean connective $*$. Then for every function $g\colon V \to S$, there is a* unique *extension $\hat{g}\colon \mathrm{PF}(V) \to S$ such that*

(1) *$\hat{g}(\neg F) = f_\neg(\hat{g}(F))$ for all F in $\mathrm{PF}(V)$;*

(2) *$\hat{g}(F * G) = f_*(\hat{g}(F), \hat{g}(G))$ for all F and G in $\mathrm{PF}(V)$ and each binary Boolean connective $*$.*

Proof. Let $\mathrm{PF}_n(V)$ be the set of elements of $\mathrm{PF}(V)$ of length at most n. Suppose h_n and h'_n have the desired properties of $\hat{g}$, but are defined only on $\mathrm{PF}_n(V)$. By induction, the set

$$\{F \in \mathrm{PF}(V) \colon h_n(F) = h'_n(F) \text{ or } F \notin \mathrm{PF}_n(V)\}$$

is just $\mathrm{PF}(V)$, so $h_n = h'_n$.

Also, in the obvious way, by Lemma 2.3, we can extend h_n to a function h_{n+1} on $\mathrm{PF}_{n+1}(V)$ having the properties of $\hat{g}$. Finally, $h_0 = \varnothing$; so h_n exists uniquely for all n in ω, and we can let $\hat{g} = \bigcup_{n\in\omega} h_n$. □

For a first application of Lemma 2.4, we can define a function

$$F \mapsto V_F \colon \mathrm{PF}(V) \to \mathcal{P}(V)$$

by the requirements:

(0) $V_P = \{P\}$ if $P \in V$;

(1) $V_{\neg F} = V_F$;

(2) $V_{F*G} = V_F \cup V_G$.

Theorem 2.5. *V_F is the set of variables that actually appear in F. In particular, if also $F \in \mathrm{PF}(V')$, then $V_F = V'_F$.*

For a second application of Lemma 2.4, letting S be the universe of $\mathbb{F}_2$, we can define

$$\begin{aligned}
f_\neg(x) &= 1 + x,\\
f_\wedge(x, y) &= xy,\\
f_\vee(x, y) &= x + y + xy,\\
f_\to(x, y) &= 1 + x + xy,\\
f_\leftrightarrow(x, y) &= 1 + x + y.
\end{aligned}$$

Letting g be a truth-assignment $\chi_{\mathfrak{A}}$, we get an extension $\hat{\chi}_{\mathfrak{A}}\colon \mathrm{PF}(V) \to 2$.

Lemma 2.6. *$\hat{\chi}_{\mathfrak{A}}(F)$ depends only on F and $A \cap V_F$, that is,*

$$\hat{\chi}_{\mathfrak{A}}(F) = \hat{\chi}_{\mathfrak{A}'}(F),$$

where $\mathfrak{A}' = (A', V')$ and $F \in \mathrm{PF}(V) \cap \mathrm{PF}(V')$, provided $A \cap V_F = A' \cap V'_F$.

If $\hat{\chi}_{\mathfrak{A}}(F) = 1$, we say that $\mathfrak{A}$ is a **model** of F and write

$$\mathfrak{A} \models F.$$

If every structure for V is a model of F, then we say F is a **validity for** V and write

$$\models_V F.$$

The **truth-table for** F is the function

$$A \mapsto \hat{\chi}_{\mathfrak{A}}(F)\colon \mathcal{P}(V_F) \to 2;$$

this is well-defined by Lemma 2.6. If this function is identically 1, then we say that F is a **tautology** and write

$$\vdash F.$$

Theorem 2.7 (Completeness). *For all F in $\mathrm{PF}(V)$,*

$$\models_V F \iff \vdash F.$$

Hence we may write the single symbol $\models$ in place of $\models_V$ and $\vdash$. If $\Phi \subseteq \mathrm{PF}(V)$, and $\mathfrak{A}$ is a model of every formula in Φ, then $\mathfrak{A}$ is a **model** of F, and we can write

$$\mathfrak{A} \models \Phi.$$

Theorem 2.8 (Compactness). *A set Φ of propositional formulas for a countable set of variables has a model if and only if each finite subset of Φ has a model.*

If every model of a set Φ of formulas is a model of some formula F, then F is a **(logical) consequence** of Φ, and we can write

$$\Phi \models F.$$

Also, $F \models G$ means $\{F\} \models G$. In this case, if also $G \models F$, then F and G are **(logically) equivalent**, and we may write

$$F \models\dashv G.$$

Lemma 2.9. *For all F and G in $\mathrm{PF}(V)$,*

$$F \models\dashv G \iff \models F \leftrightarrow G.$$

Lemma 2.10. *For all F and G in $\mathrm{PF}(V)$,*

$$\begin{aligned}
(F \vee G) &\models\dashv \neg(\neg F \wedge \neg G),\\
(F \to G) &\models\dashv (\neg F \vee G),\\
(F \leftrightarrow G) &\models\dashv ((F \to G) \wedge (G \to F)).
\end{aligned}$$

Remark 2.11. By Lemma 2.10, one can show that every equivalence-class of formulas contains a formula whose only Boolean connectives are $\neg$ and $\wedge$.

Theorem 2.12 (Adequacy). *Let A be a finite non-empty set of variables. Every function from $\mathcal{P}(A)$ to 2 is the truth-table for some propositional formula.*

Proof. We shall use induction on the size of A. If $A = \{P\}$, then the truth-tables of P, $\neg P$, $(P \wedge \neg P)$ and $(P \vee \neg P)$ are just the 4 possible functions from $\mathcal{P}(A)$ to 2.

Suppose A has size n, and P is a variable not in A, and f is a function from $\mathcal{P}(A \cup \{P\})$ to 2. Let f_0 be the restriction of f to $\mathcal{P}(A)$, and let f_1 be the function

$$B \mapsto f(B \cup \{P\})\colon \mathcal{P}(A) \to 2.$$

Then for all B in $\mathcal{P}(A \cup \{P\})$ we have

$$f(B) = \begin{cases} f_0(B \smallsetminus \{P\}), & \text{if } P \notin B; \\ f_1(B \smallsetminus \{P\}), & \text{if } P \in B. \end{cases}$$

Now let F be a formula $((F_0 \wedge \neg P) \vee (F_1 \wedge P))$, where F_0 and F_1 are formulas whose variables are from A. If $B \subseteq A \cup \{P\}$, then

$$\hat{\chi}_B(F) = \begin{cases} \hat{\chi}_{B \smallsetminus \{P\}}(F_0), & \text{if } P \notin B; \\ \hat{\chi}_{B \smallsetminus \{P\}}(F_1), & \text{if } P \in B. \end{cases}$$

Hence, if f_0 and f_1 are the truth-tables for F_0 and F_1 respectively, then f is the truth-table for F. □

Remark 2.13. We might define **propositional logic** as the use of formulas to represent functions from the power-sets of finite sets to 2. We may then say that our particular propositional logic uses the **signature** $\{\wedge, \vee, \neg, \to, \leftrightarrow\}$. The last theorem shows that this signature is **adequate** to the task of representing these functions; in fact, the theorem shows that $\{\wedge, \vee, \neg\}$ is adequate. We then have, by Remark 2.11, that the signature $\{\wedge, \neg\}$ is adequate. In fact, one could get by with a single connective, namely the binary connective $\mid$ such that

$$F \mid G \models\dashv \neg(F \wedge G);$$

this connective is called the **Sheffer stroke**, although Church in [3, n. 207, pp. 133 f.] says that Sheffer never used the stroke this way.

3 First-order logic

We now define **first-order structures** and their signatures. The structures are primary in interest, but in giving definitions, it is easier to start with signatures.

Remark 3.1. A standard example of a first-order structure is $\mathbb{R}$, considered as the 7-tuple $(R, +, -, \cdot, 0, 1, \leqslant)$, where R is the *set* of real numbers. A group is a first-order structure when considered as the ordered quadruple $(G, \cdot, {}^{-1}, 1)$; but it is *not* first-order when one considers it to be equipped also with the operation $S \mapsto \langle S \rangle$ that assigns to each subset the subgroup that it generates.

A **(first-order) signature** is a set, each of whose members can be uniquely recognized as a **function-**, **relation-** or **constant-symbol**. Each of the function- and relation-symbols has an **arity**: each of these symbols is **n-ary** for some unique positive integer n.

Let $\mathcal{L}$ be a signature. Let f, R and c be arbitrary function-, relation- and constant-symbols, respectively, of $\mathcal{L}$, and let n stand for the arity of f or R. An **$\mathcal{L}$-structure** is an ordered pair

$$(M, \mathrm{int}),$$

where M is a non-empty set, and int is a function $s \mapsto s^{\mathfrak{M}}$ on $\mathcal{L}$ such that

- $f^{\mathfrak{M}}$ is a n-ary operation on M, that is, a function from M^n to M;
- $R^{\mathfrak{M}}$ is an n-ary relation on M, that is, a subset of M^n;
- $c^{\mathfrak{M}} \in M$.

The structure itself can be denoted $\mathfrak{M}$. The set M is the **universe** of $\mathfrak{M}$, and each image $s^{\mathfrak{M}}$ is the **interpretation** of s in $\mathfrak{M}$.

Remark 3.2. A structure can be considered as its universe together with the interpretations of the symbols in its signature. This is how $\mathbb{R}$ was presented in Remark 3.1. A structure without any relations can be called an **algebra**. Theorem 2.4 involves an algebra, namely $(S, f_\wedge, f_\vee, f_\neg, f_\to, f_\leftrightarrow)$. The natural numbers compose the algebra $(\omega, ', 0)$, where $'$ is the successor-operation. The complete set of propositional formulas in some set of variables is the universe of an algebra in an obvious way.

Subsets of M^0 are **nullary** relations. There are only two of these, namely 0 and 1, which we may read as before as 'false' and 'true'.

Let X be set of new symbols, called **(individual-) variables**. We shall develop a **language**, which we might denote

$$\mathcal{L}^X.$$

The symbols of $\mathcal{L}^X$ will compose the disjoint union

$$X \cup \mathcal{L} \cup \{=\} \cup S \cup \{\exists x : x \in X\},$$

where S is an adequate signature for a propositional logic (along with brackets, if one is using them). Let us consider S to be the signature $\{\wedge, \neg\}$. Note that each $\exists x$ is an indivisible symbol, in which, however, the original x can be recognized. The symbols that are not in $X \cup \mathcal{L}$ are **logical** symbols. In every $\mathcal{L}$-structure $\mathfrak{M}$, each symbol s of $\mathcal{L}^X$ has an interpretation (rather, a family of interpretations) $s^{\mathfrak{M}}$. We have defined the interpretations of the elements of $\mathcal{L}$. The interpretations of the rest of the symbols of $\mathcal{L}^X$ are certain operations or, in one case, a relation, associated with appropriate finite subsets I of X:

- If $x \in I$, then $x^{\mathfrak{M}}$ is $\mathbf{a} \mapsto a_x : M^I \to M$.
- $=^{\mathfrak{M}}$ is equality, a subset of M^2.
- $\wedge^{\mathfrak{M}}$ is $(A, B) \mapsto A \cap B : \mathcal{P}(M^I) \times \mathcal{P}(M^I) \to \mathcal{P}(M^I)$ for any I.
- $\neg^{\mathfrak{M}}$ is $A \mapsto A^{\mathrm{c}} : \mathcal{P}(M^I) \to \mathcal{P}(M^I)$ for any I.

- $\exists x^{\mathfrak{M}}$ is, for any I, the map from $\mathcal{P}(M^I)$ to $\mathcal{P}(M^{I \smallsetminus \{x\}})$ induced by the projection $\mathbf{a} \mapsto (a_i : i \in I \smallsetminus \{x\}) : M^I \to M^{I \smallsetminus \{x\}}$.

The symbols of $\mathcal{L}^X$ compose strings of various kinds, and each of these strings has a family of interpretations. Certain strings are called **terms**, and their interpretations are functions. For each term t and for each finite set I of variables that contains the variables in t, there will be an interpretation $t^{\mathfrak{M}} : M^I \to M$. The precise definitions are thus:

- Each c is a term, and $c^{\mathfrak{M}}$, besides being an element of M, can also be understood as the constant-function $\mathbf{a} \mapsto c^{\mathfrak{M}} : M^I \to M$.
- Each variable x is a term, interpreted as above.
- If $t_0, \ldots, t_{n-1}$ are terms, then $ft_0 \ldots t_{n-1}$ is a term, with interpretation

$$\mathbf{a} \mapsto f^{\mathfrak{M}}(t_0^{\mathfrak{M}}(\mathbf{a}), \ldots, t_{n-1}^{\mathfrak{M}}(\mathbf{a})) : M^I \to M.$$

The **formulas** are certain strings whose interpretations are relations. For any **atomic** formula α and any finite set I of variables that contains the variables appearing in α, there will be an interpretation $\alpha^{\mathfrak{M}}$ that is a subset of M^I. The precise definitions are:

- If $t_0, \ldots, t_{n-1}$ are terms, then $Rt_0 \ldots t_{n-1}$ is an atomic formula, with interpretation $\{\mathbf{a} \in M^I : (t_0^{\mathfrak{M}}(\mathbf{a}), \ldots, t_{n-1}^{\mathfrak{M}}(\mathbf{a})) \in R^{\mathfrak{M}}\}$.
- If t_0 and t_1 are terms, then $t_0 = t_1$ is an atomic formula, with interpretation $\{\mathbf{a} \in M^I : t_0^{\mathfrak{M}}(\mathbf{a}) = t_1^{\mathfrak{M}}(\mathbf{a})\}$.

The formulas in general are built up using the remaining logical symbols: The atomic formulas are formulas, and if ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $\neg\phi$ and $\exists x\ \phi$ for any x in X. The interpretations are obvious:

- $(\phi \wedge \psi)^{\mathfrak{M}} = \phi^{\mathfrak{M}} \cap \psi^{\mathfrak{M}}$;
- $\neg\phi^{\mathfrak{M}} = (\phi^{\mathfrak{M}})^{\mathrm{c}}$;
- $\exists x\ \phi^{\mathfrak{M}} = \exists x^{\mathfrak{M}}(\phi^{\mathfrak{M}})$.

In particular, if $\phi^{\mathfrak{M}}$ is a well-defined subset of M^I, then $\exists x\ \phi^{\mathfrak{M}}$ is a well-defined subset of $M^{I \smallsetminus \{x\}}$. Thus, the nullary relations 0 and 1 can arise as interpretations. To say precisely when they can arise, we recursively define the set $\mathrm{FV}(\phi)$ of **free variables** of an arbitrary formula ϕ:

- $\mathrm{FV}(\alpha)$ is the set of variables appearing in α, if α is atomic;
- $\mathrm{FV}(\neg\phi) = \mathrm{FV}(\phi)$;
- $\mathrm{FV}(\phi \wedge \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$;
- $\mathrm{FV}(\exists x\ \phi) = \mathrm{FV}(\phi) \smallsetminus \{x\}$.

Then $\phi^{\mathfrak{M}}$ is defined as a subset of M^I, provided $\mathrm{FV}(\phi) \subseteq I$.

Theorem 3.3 (Substitution). *Suppose the following:*

- *ϕ is a formula of $\mathcal{L}^X$;*
- *I is a finite subset of X such that* $\mathrm{FV}(\phi) \subseteq I$*;*
- *$\mathbf{u}$ is an I-tuple of terms of $\mathcal{L}^X$;*
- *J is a finite subset of X that contains all variables in the entries in $\mathbf{u}$.*

Then there is a formula $\phi(\mathbf{u})$ of $\mathcal{L}^X$ such that $\mathrm{FV}(\phi(\mathbf{u})) \subseteq J$ *and, for every $\mathcal{L}$-structure $\mathfrak{M}$, and for all $\mathbf{a}$ in M^J,*

$$\mathbf{a} \in \phi(\mathbf{u})^{\mathfrak{M}} \iff (u_i^{\mathfrak{M}}(\mathbf{a}) : i \in I) \in \phi^{\mathfrak{M}}.$$

Example 3.4. If $\mathrm{FV}(\phi) \subseteq I$, and $\mathbf{x}$ is the identity on I (so $\mathbf{x} = (x : x \in I)$), then we may assume that $\phi(\mathbf{x})$ is the same formula as $\mathbf{x}$.

A **sentence** is a formula with no free variables. If σ is a sentence of $\mathcal{L}^X$, and $\sigma^{\mathfrak{M}} = 1$, then we say that $\mathfrak{M}$ is a **model** of σ.

Example 3.5. If $\mathrm{FV}(\phi) \subseteq I$, and $\mathbf{a}$ is an I-tuple of constant-symbols, then $\phi(\mathbf{c})$ is a sentence σ such that

$$\sigma^{\mathfrak{M}} = 1 \iff \mathbf{c}^{\mathfrak{M}} \in \phi^{\mathfrak{M}},$$

where $\mathbf{c}^{\mathfrak{M}}$ is $(c_i^{\mathfrak{M}} : i \in I)$.

We can also allow structures to be models of arbitrary formulas. Suppose ϕ is a formula of $\mathcal{L}^X$ and $\mathrm{FV}(\phi) \subseteq I$. If $\mathbf{c}$ is an I-tuple of constant-symbols that are *not* in $\mathcal{L}$, and $\mathbf{a}$ is an I-tuple from M, then $(\mathfrak{M}, \mathbf{a})$ is a structure of $\mathcal{L} \cup \{c_x : x \in I\}$ in the obvious way. Then $\mathfrak{M}$ is a model of ϕ, provided $(\mathfrak{M}, \mathbf{a})$ is a model of $\phi(\mathbf{c})$ for *some* tuple $\mathbf{a}$.

The notations of § 2 involving $\models$ now make sense in the present context. If $\mathfrak{M} \models \phi$, we say also that $\mathfrak{M}$ **satisfies** ϕ.

We may let $\mathcal{L}(M)$ be the disjoint union $\mathcal{L} \sqcup M$, where each element of M is understood as a constant-symbol whose interpretation in $\mathfrak{M}$ is itself. Then we may ask whether $\mathfrak{M}$ is a model of a formula of $\mathcal{L}(M)^X$.

In the following, $\exists \mathbf{x}$ is an abbreviation for

$$\exists x_0 \, \exists x_1 \, \ldots \exists x_{n-1} ,$$

where $I = \{x_0, \ldots, x_{n-1}\}$.

Lemma 3.6. *Suppose $\mathfrak{M}$ is an $\mathcal{L}$-structure, and* $\mathrm{FV}(\phi) \subseteq I$*. The following are equivalent:*

(0) $\mathfrak{M}$ satisfies ϕ;

(1) $\mathfrak{M} \models \phi(\mathbf{a})$ for some $\mathbf{a}$ in M^I;

(2) $\mathfrak{M} \models \exists \mathbf{x} \, \phi(\mathbf{x})$.

4 Types

Now suppose that the set X of individual-variables is $\{x_i : i \in \omega\}$, and write $\mathcal{L}^X$ just as $\mathcal{L}$. On the set of formulas of $\mathcal{L}$ with free variables in $\{x_i : i < n\}$, the relation $\models \dashv$ is an equivalence-relation; let us denote the set *modulo* the relation by

$$\mathrm{Fm}^n(\mathcal{L}).$$

Then $\wedge$ and $\neg$ (and hence $\vee$) are well-defined operations on $\mathrm{Fm}^n(\mathcal{L})$, which is also partially ordered by $\models$ and, with respect to this, has a greatest element $\top$ and a least element $\bot$.

An n**-type of** $\mathcal{L}$ is a subset Γ of $\mathrm{Fm}^n(\mathcal{L})$ such that

- $\phi, \psi \in \Gamma \implies \phi \wedge \psi \in \Gamma$;
- $\phi \in \Gamma \;\&\; \phi \models \psi \implies \psi \in \Gamma$;
- $\top \in \Gamma$;

the type is **proper** if $\bot \notin \Gamma$; if proper, the type is **complete** if

- $\phi \in \mathrm{Fm}^n(\mathcal{L}) \smallsetminus \Gamma \implies \neg\phi \in \Gamma$.

The unique **improper** n-type is $\mathrm{Fm}^n(\mathcal{L})$ itself. Every subset of $\mathrm{Fm}^n(\mathcal{L})$ generates a type, possibly improper. The subset itself can be called **consistent** or **finitely satisfiable** if for every finite subset $\{\phi_i : i < m\}$ there is a structure satisfying $\bigwedge_{i<m} \phi_i$.

Lemma 4.1. *A subset of* $\mathrm{Fm}^n(\mathcal{L})$ *is finitely satisfiable if and only if it generates a proper type.*

In fact the structure $(\mathrm{Fm}^n(\mathcal{L}), \wedge, \vee, \neg, \bot, \top, \models)$ is a *Boolean algebra*. A standard Boolean algebra is

$$(\mathcal{P}(X), \cap, \cup, {}^{\mathrm{c}}, \varnothing, X, \subseteq),$$

where X is a set. One way to give a formal definition is the following. A **Boolean ring** is a (unital, associative) ring satisfying $\forall x\; x \cdot x = x$.

Lemma 4.2. *Boolean rings are commutative and are of characteristic* 2.

Example 4.3. $\mathbb{F}_2$ is a Boolean ring.

Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, and define new operations and a relation on B by the following rules (which should be compared with the definition of $\hat{\chi}_{\mathfrak{A}}$ on p. 4):

- $x \wedge y = xy$;
- $x \vee y = x + y + xy$;
- $\neg x = 1 + x$;
- $x \leqslant y \iff x \wedge y = x$;
- $\bot = 0$ and $\top = 1$.

The structure $(B, \wedge, \vee, \neg, \bot, \top, \leqslant)$ arising thus is a **Boolean algebra**.

Lemma 4.4. *If $(B, \wedge, \vee, \neg, \bot, \top, \leqslant)$ is a Boolean algebra, then the Boolean ring from which it arises is given by*

- $xy = x \wedge y$;
- $x + y = \neg(\neg x \wedge \neg y) \wedge \neg(x \wedge y)$;
- $\top = 1$ *and* $\bot = 0$.

Lemma 4.5. $\mathrm{Fm}^n(\mathcal{L})$ *is a Boolean algebra.*

A subset F of a Boolean algebra is a **filter** if the set $\{x : \neg x \in F\}$ is an ideal of the corresponding ring; F is **principal** if I is principal; F is an **ultra-filter** if I is maximal. (The unique improper filter is the algebra itself; an ultra-filter must be proper.)

Lemma 4.6. *Types of* $\mathrm{Fm}^n(\mathcal{L})$ *are just filters; complete types are just ultra-filters.*

The set of ultra-filters of a Boolean algebra $\mathfrak{B}$ is denoted

$$\mathrm{S}(\mathfrak{B})$$

and called its **Stone space**, because of the following. If $x \in B$, let

$$[x]$$

be the subset $\{F : x \in F\}$ of $\mathrm{S}(\mathfrak{B})$.

Theorem 4.7 (Stone Representation). *If $\mathfrak{B}$ be a Boolean algebra, then the map*

$$x \mapsto [x] : \mathfrak{B} \to \mathcal{P}(\mathrm{S}(\mathfrak{B}))$$

is an embedding of Boolean algebras.

Corollary 4.8. *The subsets $[x]$ of $\mathrm{S}(\mathfrak{B})$ compose a basis of open sets and of closed sets for a topology on $\mathrm{S}(\mathfrak{B})$, which topology is compact and Hausdorff.*

For every subset X of B, let $\overline{X}$ be the subset $\bigcap_{x \in X}[x]$ of $\mathrm{S}(\mathfrak{B})$.

Lemma 4.9. *Suppose $\mathfrak{B}$ is a Boolean algebra.*

(0) The map

$$X \mapsto \overline{X} : \mathcal{P}(B) \to \mathcal{P}(\mathrm{S}(\mathfrak{B}))$$

is inclusion-reversing and takes unions to intersections, and its range is the set of closed subsets of $\mathrm{S}(\mathfrak{B})$.

(1) The map

$$Y \mapsto \bigcap Y : \mathcal{P}(\mathrm{S}(\mathfrak{B})) \to \mathcal{P}(\mathfrak{B})$$

is inclusion-reversing and takes unions to intersections, and its range is the set of filters of $\mathfrak{B}$.

(2) If $X \subseteq B$, then $\bigcap \overline{X}$ is the filter of $\mathfrak{B}$ generated by X.

(3) If $Y \subseteq \mathrm{S}(\mathfrak{B})$, then $\overline{\bigcap Y}$ is the topological closure of Y.

hence $X \mapsto \overline{X}$ gives a one-to-one correspondence, with inverse $Y \mapsto \bigcap Y$, between filters of $\mathfrak{B}$ and closed subsets of $\mathrm{S}(\mathfrak{B})$.

So the complete n-types of $\mathcal{L}$ compose a compact Hausdorff space, denoted

$$\mathrm{S}^n(\mathcal{L}),$$

whose closed subsets are just the sets $\overline{\Gamma}$ determined by arbitrary n-types Γ.

A **theory** of $\mathcal{L}$ is a 0-type The improper 0-type of $\mathcal{L}$ is the unique **inconsistent** theory of $\mathcal{L}$.

Since $\mathrm{Fm}^0(\mathcal{L})$ embeds in $\mathrm{Fm}^n(\mathcal{L})$, a theory T of $\mathcal{L}$ determines a closed subset of $\mathrm{S}^n(\mathcal{L})$, denoted

$$\mathrm{S}^n(T).$$

Then an arbitrary n-type Γ is **consistent with** T if $\Gamma \cup T$ is consistent, equivalently, $\overline{\Gamma} \cap \mathrm{S}^n(T) \neq 0$.

Theorem 4.10 (Compactness). *Every consistent theory has a model.*

Proof. Let T be a theory of $\mathcal{L}$. The proof that T has a model has three parts:

(0) There is a signature $\mathcal{L}'$ such that $\mathcal{L} \subseteq \mathcal{L}'$, and $\mathcal{L}' \smallsetminus \mathcal{L}$ consists of constant-symbols, and there is a bijection

$$\phi \mapsto c_\phi : \mathrm{Fm}^1(\mathcal{L}') \to \mathcal{L}' \smallsetminus \mathcal{L}.$$

Now let $H(T)$ be the set $\mathrm{S}^0(T) \cap \bigcap_{\phi \in \mathrm{Fm}^1(\mathcal{L}')} [\exists x_0\ \phi \to \phi(c_\phi)]$.

(1) Let T' be an element of $H(T)$. Then T' has a **canonical model**, whose universe is $\mathcal{L}' \smallsetminus \mathcal{L}$ *modulo* the equivalence-relation $\sim$ given by

$$c \sim d \iff T' \models c = d.$$

(2) $H(T)$ is non-empty.

Note that $H(T)$ is non-empty by Corollary 4.8, in particular, compactness of $\mathrm{S}^0(T)$. □

If Γ is an n-type, and $\mathbf{c}$ is an n-tuple of constant-symbols, then the set $\{\phi(\mathbf{c}) : \phi \in \Gamma\}$ can be denoted

$$\Gamma(\mathbf{c}).$$

A structure $\mathfrak{M}$ **realizes** Γ if $\mathfrak{M} \models \Gamma(\mathbf{a})$ for some tuple $\mathbf{a}$ from M; otherwise the structure **omits** the type.

An complete type p is: **isolated**, if $\{p\}$ is open; **limit**, if not. These definitions can be understood absolutely, as stated, or **over** some theory T.

Lemma 4.11. *The isolated types are precisely the principal complete types. Every type included in a principal type over a* complete *theory T is realized in every model of T. If Γ is a type consistent with an (arbitrary) theory T, then the following are equivalent:*

(0) Γ is not included in a principal type over T;

(1) $\overline{\Gamma}$ *has empty interior in* $\mathrm{S}^n(T)$*;*

(2) $\overline{\Gamma}^{\mathrm{c}}$ *is a dense open subset of* $\mathrm{S}^n(T)$*.*

Example 4.12. Let $\mathcal{L}$ be $\{c_n : n \in \omega\} \cup \{P\}$, where the c_n are constant-symbols and P is a unary relation-symbol. Let T be the theory generated by $\{Pc_n : n \in \omega\}$. Then

$$T \models \neg Px \to x \neq c_n$$

for each n, so the principal type generated by $\neg Px$ includes the type generated by $\{x \neq c_n : n \in \omega\}$; but the latter type is not principal.

A partial converse is the Omitting-Types Theorem below, whose proof is based on [7, ch. 10]. First:

Lemma 4.13. *The intersection of countably dense open subsets of a compact Hausdorff space is also dense.*

Proof. Suppose X is a compact Hausdorff space. Then X is locally compact, that is, every neighborhood of every point includes a compact neighborhood. Indeed, let U be an open neighborhood of P. For each x in U^{c} there are disjoint open neighborhoods V_x and U_x of x and P respectively. Some finite union of sets V_x covers U^{c}; the complement is included in U and is a closed—hence compact—neighborhood of P, since it includes the corresponding intersection of sets U_x.

Now suppose $\{O_n : n \in \omega\}$ is a collection of dense open subsets of X. We can recursively define a decreasing chain $U_0 \supseteq K_0 \supseteq U_1 \supseteq K_1 \supseteq U_2 \supseteq \dots$ of sets, and at the same time a sequence $(P_n : n \in \omega)$ of points, such that:

- $U_0 = U$;
- U_n is open;
- $P_n \in U_n \cap O_n$;
- K_n is compact, and $P_n \in K_n \subseteq U_n \cap O_n$;
- $P_n \in U_{n+1} \subseteq K_n$.

Then $\bigcap_{n\in\omega} K_n$ is a nonempty subset of U included in each set O_n. □

Theorem 4.14 (Omitting Types). *Let T be a consistent theory of a countable signature $\mathcal{L}$. For every countable collection of types Γ, none included in a principal type, T has a countable model omitting each Γ.*

Proof. To the proof of the Compactness Theorem, we add a step:

(3) $H(T)$ has an element T' such that, for each tuple $\mathbf{c}$ of elements of $\mathcal{L}' \smallsetminus \mathcal{L}$, and for each Γ,

$$T' \notin \overline{\Gamma(\mathbf{c})},$$

that is, $\Gamma(\mathbf{c}) \not\subseteq T'$.

To prove this, by Lemma 4.13, it is enough to show that each closed set $\overline{\Gamma(\mathbf{c})}$ has dense complement in $H(T)$, since then the intersection of these complements is dense and so non-empty.

Every open subset of $H(T)$ is a union of sets $[\psi(\mathbf{d})] \cap H(T)$, where ψ is a formula of $\mathcal{L}$, and $\mathbf{d}$ is a tuple of elements of $\mathcal{L}' \smallsetminus \mathcal{L}$. Supposing

$$T' \in [\psi(\mathbf{d})] \cap H(T),$$

we shall derive an element T^* of $[\psi(\mathbf{d})] \cap H(T) \smallsetminus \overline{\Gamma(\mathbf{c})}$.

Each entry of $(\mathbf{c}, \mathbf{d})$ is c_{ϕ^0} for some formula ϕ^0, which contains finitely many constant-symbols c_{ϕ^1}; each ϕ^1 contains finitely many constant-symbols c_{ϕ^2}, and so on. The constant-symbols arising in this way form a finitely branching tree with no infinite branches; hence they are finitely numerous and compose a tuple $\mathbf{e}$.

Hence if c_ϕ is one of the terms of $(\mathbf{c}, \mathbf{d}, \mathbf{e})$, then the constant-symbols used in ϕ also appear in $(\mathbf{c}, \mathbf{d}, \mathbf{e})$. Hence there is a formula θ of $\mathcal{L}$ such that $\theta(\mathbf{c}, \mathbf{d}, \mathbf{e})$ is the conjunction of $\psi(\mathbf{d})$ and the sentences

$$\exists x_0 \phi \to \phi(c_\phi)$$

such that c_ϕ appears in $(\mathbf{c}, \mathbf{d}, \mathbf{e})$.

Let $\mathfrak{M}$ be the canonical model of T'. Then $\mathfrak{M} \models \theta(\mathbf{c}, \mathbf{d}, \mathbf{e})$. The open set $[\exists \mathbf{y}\ \exists \mathbf{z}\ \theta(\mathbf{x}, \mathbf{y}, \mathbf{z})]$ is therefore a non-empty subset of $\mathrm{S}^n(T)$, so it is not included in $\overline{\Gamma}$. Suppose

$$p \in [\exists \mathbf{y}\ \exists \mathbf{z}\ \theta(\mathbf{x}, \mathbf{y}, \mathbf{z})] \smallsetminus \overline{\Gamma}.$$

By the Compactness Theorem, T has a countable model $\mathfrak{N}$ realizing p with some tuple $\mathbf{a}$. There is a bijection f between $\mathcal{L}' \smallsetminus \mathcal{L}$ and N such that $f(\mathbf{c}) = \mathbf{a}$ and

$$\mathfrak{N} \models \theta(f(\mathbf{c}, \mathbf{d}, \mathbf{e})).$$

This bijection determines the desired T^*. □

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Hints

- Lemma 2.4: Prove that for each n there is a unique such function on the set of formulas of length at most n.
- Theorem 2.7: It's a simple chain of equivalences, justified by Lemma 2.6.
- Theorem 2.8: Say $V = \{P_n : n \in \omega\}$. Define $V_n = \{P_i : i < n\}$. Let T be the set of structures on the various V_n. Order T by the rule

  $$(A, V_m) \leqslant (B, V_n) \iff m \leqslant n \land A = V_m \cap B.$$

  Then $(T, \leqslant)$ is a tree. Consider the set comprising those (A, V_m) such that, for all F in Φ, if $V_F \subseteq V_m$, then $(A, V_m) \models F$. This set forms an infinite sub-tree of T. Hence the sub-tree includes an infinite chain.
- Lemma 2.9: Use $f_{\leftrightarrow}$.
- Lemma 2.10: Use the various f_*.