Homework IV, Math 736, Model-Theory.

Elements $a_0, \ldots, a_{k-1}$ of an abelian group are called **additively independent** if

$$\sum_{i<k} n_i a_i \neq 0$$

for all integers $n_0 \ldots, n_{k-1}$, not all of which are 0.

Problem 7. *Let* $\mathbf{Q}$ *be the* abelian group *of rational numbers. Show that there is an abelian group* $\mathcal{G}$ *such that:*

(∗) $\mathcal{G} \equiv \mathbf{Q}$, *and*

(†) $\mathcal{G}$ *contains* n *additively independent elements for every* n *in* ω.

Then show that any two such countable *groups are isomorphic.*

Now let $\mathcal{L}$ be an arbitrary signature. If $\mathcal{M}, \mathcal{N} \in \mathfrak{Mod}(\mathcal{L})$ and $\mathcal{M} \subseteq \mathcal{N}$, let us write

$$\mathcal{M} \preccurlyeq_1 \mathcal{N}$$

if the inclusion of M in N preserves *universal* formulas of $\mathcal{L}$.

Problem 8. *Prove that the following are equivalent:*

(∗) $\mathcal{M} \preccurlyeq_1 \mathcal{N}$

(†) *there is* $\mathcal{R}$ *in* $\mathfrak{Mod}(\mathcal{L})$ *such that* $\mathcal{M} \preccurlyeq \mathcal{R}$ *and* $\mathcal{N} \subseteq \mathcal{R}$.

Suppose $\{\mathcal{M}_n : n \in \omega\}$ is a subset of $\mathfrak{Mod}(\mathcal{L})$ forming a **chain**, that is, $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$ for all n in ω. Then the **union** of this chain is defined to be the structure $\mathcal{N}$, where:

(∗) $N = \bigcup_{n\in\omega} M_n$, and

(†) for all basic formulas ϕ, if $\mathbf{a}$ is a tuple from M_n, and $\mathcal{M}_n \models \phi(\mathbf{a})$, then $\mathcal{N} \models \phi(\mathbf{a})$.

(You should verify that $\mathcal{N}$ is well-defined, but you need not submit the verification.) The chain is called **elementary** if $\mathcal{M}_n \preccurlyeq \mathcal{M}_{n+1}$ for all n.

Problem 9. *Show that the union of an elementary chain is an elementary extension of each structure in the chain.*

Recall that a theory T of $\mathcal{L}$ is called **model-complete** if $\mathcal{M} \preccurlyeq \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and both structures are models of T. A formally weaker notion is **1-model-completeness**: T is 1-model-complete if $\mathcal{M} \preccurlyeq_1 \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and both structures are models of T.

Problem 10. *Prove that 1-model-completeness and model-completeness coincide.*