Model-Theory to Compactness

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October 26, 2001

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0 Introduction

These notes are an attempt to develop model theory, as economically as possible, on a foundation of some familiarity with algebraic structures. (Formal definitions of these structures are given in § 6.) References for model-theory include [1], [2] and [3].

Words in **boldface** are technical terms and are often being defined, implicitly or explicitly, by the sentence in which they occur.

1 The natural numbers

By one standard definition, the set ω of **natural numbers** is the smallest set that contains the empty set and that contains $x \cup \{x\}$ whenever it contains x. The empty set will be denoted 0 here, and $x \cup \{x\}$, the **successor** of x, can be denoted x'. The triple $(\omega, ', 0)$ will turn out to be an example of a *structure*.

Throughout these notes, n will be a natural number, understood as the set $\{0, 1, 2, \ldots, n-1\}$, possibly empty; and i will range over the elements of this set. Also m will be a natural number.

2 Cartesian powers

Let M be a set. The **Cartesian power** M^n is the set of functions from n to M. Such a function will be denoted by a boldface letter, as $\mathbf{a}$, but then its value $\mathbf{a}(i)$ at i will be denoted a_i. The function $\mathbf{a}$ can be identified with the n**-tuple** $(a_0, \dots, a_{n-1})$ of its values.

In particular, the power M^0 has but a single member, $()$ or 0; hence $M^0 = 1$. This is so, even if $M = 0$; however, $0^n = 0$ when n is *positive* (different from 0). The set M itself can be identified with the power M^1.

Any function $f : m \to n$ determines the map

$$\mathbf{a} \mapsto (a_{f(0)}, \dots, a_{f(m-1)}) : M^n \to M^m,$$

no matter what set M is. In case $m = 1$, we have the **coordinate projections** $\mathbf{a} \mapsto a_i$.

The **Cartesian product** $A \times B$ of sets A and B is identified with the set of (ordered) pairs (a, b) such that $a \in A$ and $b \in B$. There is a map

$$\begin{aligned} M^n \times M^m &\longrightarrow M^{n+m} \\ (\mathbf{a}, \mathbf{b}) &\longmapsto (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1}), \end{aligned}$$

often considered an identification.

3 Structures and signatures

A **function on** the set M is a map $M^n \to M$; the function then is n**-ary**—its **arity** is n. A nullary (that is, 0-ary) function is a **constant** and can be identified with an element of M.

An n**-ary relation on** M is a subset of M^n. There are two nullary relations, namely 0 and 1. The relation of *equality* is binary (2-ary).

A **structure** is a set equipped with some distinguished constants and with some functions and relations of various positive arities. The set then is the **universe** of the structure. If the universe is M, then the structure might be denoted $\mathcal{M}$ or just M again. However, the structure $(\omega, {}', 0)$ is denoted $\mathbf{N}$. (This structure is often considered to contain the binary functions of addition and multiplication as well, but these are uniquely determined by the successor-function.)

Examples. A set with no distinguished relations, functions or constants is trivially a structure. Groups, rings and partially ordered sets are structures. A vector space is a structure whose unary functions are the multiplications by the scalars. A valued field can be understood as a structure when the valuation ring is distinguished as a unary relation.

The **signature** of a structure contains a **symbol** for each function, relation and constant in the structure; the function, relation or constant is then the **interpretation** of the symbol. Notationally, the symbols are primary; their interpretations can be distinguished, if need be, by superscripts indicating the structure.

Examples. The complete ordered field $\mathbf{R}$ has the signature $\{+, -, \cdot, \leqslant, 0, 1\}$. The ordered field $\mathbf{Q}$ of rational numbers has the same signature. The binary function-symbol $+$ is interpreted in $\mathbf{R}$ by addition of real numbers; the interpretation is also denoted by $+$, or by $+^{\mathbf{R}}$ if it should be distinguished from addition $+^{\mathbf{Q}}$ of rational numbers. To make its signature explicit, we can write $\mathbf{R}$ as the tuple $(\mathbf{R}, +, -, \cdot, \leqslant, 0, 1)$; in the latter notation, we can understand $\mathbf{R}$ as the *set* of real numbers.

A structure in a given signature, say $\mathcal{L}'$, can be understood as a structure with a smaller signature, say $\mathcal{L}$: just ignore the interpretations of the symbols in $\mathcal{L}' - \mathcal{L}$. The structure in $\mathcal{L}$ is then a **reduct** of the structure in $\mathcal{L}'$, which is in turn an **expansion** of the structure in $\mathcal{L}$.

Example. The abelian group $(\mathbf{R}, +, -, 0)$ is a reduct of the ordered field $(\mathbf{R}, +, -, \cdot, \leqslant, 0, 1)$; the group can be *expanded* to the ordered field.

Throughout these notes, $\mathcal{L}$ will be a signature, and f, R and c will range respectively over the function-, relation- and constant-symbols in $\mathcal{L}$. The structures with signature $\mathcal{L}$ compose the class $\mathfrak{Mod}(\mathcal{L})$.

4 Homomorphisms and embeddings

Suppose $\mathcal{M}$ and $\mathcal{N}$ are in $\mathfrak{Mod}(\mathcal{L})$, and h is a map $M \to N$. (So, N must be nonempty, unless M is empty.) Then h induces maps $M^n \to N^n$ in the obvious way, even when $n = 0$; so, $h(\mathbf{a})(i) = h(a_i)$, and $h(0) = 0$. The map h is a **homomorphism** from $\mathcal{M}$ to $\mathcal{N}$ if it *preserves* the functions, relations and constants symbolized in $\mathcal{L}$, that is,

- $h(f^{\mathcal{M}}(\mathbf{a})) = f^{\mathcal{N}}(h(\mathbf{a}))$;
- $h(\mathbf{a}) \in R^{\mathcal{N}}$ when $\mathbf{a} \in R^{\mathcal{M}}$;
- $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

Any map preserves *equality.* A homomorphism is an **embedding** if it preserves both inequality and the complements of the relations symbolized in $\mathcal{L}$. In particular, the underlying map of an embedding is injective (or *one-to-one*); if it is also surjective (or *onto*), then the embedding is an **isomorphism**.

We may confuse a structure with its isomorphism-class.

Examples. A group-homomorphism is a homomorphism of groups; a group-monomorphism is an embedding of groups; a group-isomorphism is an isomorphism of groups.

If $M \subseteq N$, and the inclusion-map of M in N is an embedding of $\mathcal{M}$ in $\mathcal{N}$, then we write

$$\mathcal{M} \subseteq \mathcal{N}$$

and say that $\mathcal{M}$ is a **substructure** of $\mathcal{N}$.

Example. A subgroup of a group is a substructure of a group, and in fact any substructure of a group is a subgroup. However, while $\mathbf{Z}$ is a substructure of $\mathbf{R}$, it is not a subfield (because it is not a field).

5 Functions and terms

Suppose $\mathcal{M}$ is in $\mathfrak{Mod}(\mathcal{L})$. Various functions on M can be derived, by composition, from:

- the functions $f^{\mathcal{M}}$,
- the constants $c^{\mathcal{M}}$, and
- the coordinate projections.

These compositions can be described without reference to $\mathcal{M}$; the result is the **terms** of $\mathcal{L}$.

The **interpretation** $t^{\mathcal{M}}$ in $\mathcal{M}$ of an n-ary term t of $\mathcal{L}$ will be an n-ary function on M. Terms can be defined as strings of symbols so that the following hold:

- Each constant-symbol c is also an n-ary term whose interpretation in $\mathcal{M}$ is the constant map $\mathbf{a} \mapsto c^{\mathcal{M}}$ on M^n.

- There is an n-ary term x_i whose interpretation in $\mathcal{M}$ is the coordinate projection $\mathbf{a} \mapsto a_i$ on M^n.
- If $t_0, \dots, t_{n-1}$ are m-ary terms, and f is n-ary, then there is an m-ary term—call it $f(t_0, \dots, t_{n-1})$—whose interpretation is the map
$$\mathbf{a} \mapsto f^{\mathcal{M}}(t_0^{\mathcal{M}}(\mathbf{a}), \dots, t_{n-1}^{\mathcal{M}}(\mathbf{a})).$$

By this account, an n-ary term is also $n+1$-ary. The nullary terms are the **constant** terms; the terms x_i are the **variables**.

Lemma. *If t is an n-ary term, and $u_0, \dots, u_{n-1}$ are m-ary terms, then there is an m-ary term whose interpretation in $\mathcal{M}$ is the map*
$$\mathbf{a} \mapsto t^{\mathcal{M}}(u_0^{\mathcal{M}}(\mathbf{a}), \dots, u_{n-1}^{\mathcal{M}}(\mathbf{a})).$$

The new term in the lemma can of course be denoted $t(u_0, \dots, u_{n-1})$.

We can identify terms whose interpretations are indistinguishable in every structure. In particular, if t is n-ary, but not $(n-1)$-ary, then t is precisely $t(x_0, \dots, x_{n-1})$, which we may abbreviate as $t(\mathbf{x})$. Sometimes letters like x, y and z are used for variables.

If A is a subset of M, we let $\mathcal{L}(A)$ be the signature $\mathcal{L}$ augmented with a constant-symbol for each element of A. The symbols and the elements are generally not distinguished notationally, and an $\mathcal{L}$-structure $\mathcal{M}$ naturally determines an $\mathcal{L}(A)$-structure, denoted $\mathcal{M}_A$ if there is a need to distinguish.

Lemma. *Every term of $\mathcal{L}(A)$ is $t(\mathbf{a}, \mathbf{x})$ for some term t of $\mathcal{L}$ and tuple $\mathbf{a}$ from A.*

6 Algebras

Suppose $\mathcal{M} \in \mathfrak{Mod}(\mathcal{L})$. An **equation**
$$t = u$$
of n-ary terms of $\mathcal{L}$ is an **identity** of $\mathcal{M}$ if $t^{\mathcal{M}} = u^{\mathcal{M}}$; we can then write
$$\mathcal{M} \models t = u$$
and say that $\mathcal{M}$ is a **model** of $t = u$ or that $\mathcal{M}$ **satisfies** the identity.

Suppose $\mathcal{L}$ contains no relation-symbols. An element of $\mathfrak{Mod}(\mathcal{L})$ can be called an **algebra**. A set of equations of terms of $\mathcal{L}$ determines a **variety** of $\mathcal{L}$ (namely the subclass of $\mathfrak{Mod}(\mathcal{L})$ comprising each structure that is a model of each equation.) A substructure of an element of a variety is also in the variety.

Several standard classes of mathematical structures are varieties or subclasses of these, in signatures comprising some of:

0. the constant-symbols 0 and 1, for **zero** and **one**;
1. the unary function-symbols $-$ and $^{-1}$, for **additive** and **multiplicative inversion**;
2. the binary function-symbols $+$ and $\cdot$, for **addition** and **multiplication**.

Specific signatures involving these symbols are sometimes named thus:

The set:	... is the signature of:
$\{\cdot\}$	semi-groups
$\{\cdot, 1\}$	monoids
$\{\cdot, {}^{-1}, 1\}$	groups
$\{+, -, 0\}$	abelian groups
$\{+, -, \cdot, 0, 1\}$	rings

The corresponding structures will be defined presently. First, terms with these symbols are customarily written so that:

- 0, 1 and the variables x_i are terms;
- if t is a term, then so are $(-t)$ and t^{-1};
- if t and u are terms, then so are $(t+u)$ and $(t \cdot u)$.

Abbreviations of terms are also customary, so that, for example:
outer brackets can be removed;
tu means $t \cdot u$;
$t - u$ means $t + -u$;
$t * u * v$ means $((t * u) * v)$, where each $*$ is the same symbol $+$ or $\cdot$; and
$t + uv$ means $t + (uv)$.

A **semi-group** is a model of the identity

$$x(yz) = xyz.$$

Examples. The empty set is the universe of a semi-group. The structure $(M, \frown)$ is a semi-group, where M comprises the strings

$$|| \cdots |$$

consisting of some (positive, finite) number of strokes, and $\frown$ is concatenation of strings.

A **monoid** is a semi-group satisfying the identities

$$x \cdot 1 = x,$$
$$1 \cdot x = x.$$

Examples. Let M comprise the functions from some set to itself; let $\circ$ be functional composition; and let id be the identity-function on M. Then $(M, \circ, \mathrm{id})$ is a monoid. So is $(\omega, +, 0)$.

A **group** is a monoid satisfying

$$x \cdot x^{-1} = 1,$$
$$x^{-1} \cdot x = 1.$$

The group is **abelian** if it satisfies $xy = yx$—though, as noted, an abelian group is usually 'written additively,' with the signature $\{+, -, 0\}$.

Examples. The group $(\mathbf{Z}, +, -, 0)$ of integers is abelian; so is the group $(S, \cdot, ^{-1}, 1)$, where S is the circle $\{z \in \mathbf{C} : |z| = 1\}$, comprising the complex numbers of modulus 1.

A **ring** is a structure $(R, +, -, \cdot, 0, 1)$ such that:

- $(R, +, -, 0)$ is an abelian group;
- $(R, \cdot, 1)$ is a monoid;
- the identities $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$ are satisfied.

By this definition, there is a ring, the **trivial** ring, satisfying $0 = 1$, but its universe comprises a unique element.

A ring is **commutative** if it satisfies $xy = yx$. In a commutative ring, an element a is called:

- a **zero-divisor** if $a \neq 0$, but $ab = 0$ for some non-zero b in the ring;
- a **unit** if $ab = 1$ for some b in the ring.

Then a zero-divisor cannot be a unit, and zero is a unit only in the trivial ring. The set of units of a commutative ring R is denoted

$$R^{\times}.$$

Then $(R^{\times}, \cdot, 1)$ is a (well-defined) monoid and can be expanded to a group. A non-trivial ring is an **integral domain** if it is commutative and contains no zero-divisors.

Henceforth in this section, let *ring* mean *non-trivial commutative ring*, and let $(R, +, -, \cdot, 0, 1)$ be such a ring. Then R is an integral domain just in case $(R - \{0\}, \cdot, 1)$ is a well-defined monoid. If this monoid can be expanded to a group, then R is a **field**. Hence R is a field just in case $R^{\times}$ comprises all non-zero elements of R.

Examples. The sets $\mathbf{Q}$, of rational numbers; $\mathbf{R}$, of real numbers; and $\mathbf{C}$, of complex numbers—each is the universe of a field. So is their subset $\{0, 1\}$ (though the resulting field is not a substructure of these). Over any field K can be formed the **polynomial-ring**

$$K[x_0, \dots, x_{n-1}],$$

which can be defined as follows. First say that n-ary terms t and u of $\mathcal{L}(K)$ are equivalent if the identity $t = u$ is satisfied in every field of which K is a substructure. (If K is infinite, it is enough that $t^K = u^K$.) Then $K[x_0, \dots, x_{n-1}]$ comprises the equivalence-classes of the n-ary terms of $\mathcal{L}(K)$.

The signature of R**-modules** is the signature of abelian groups, with a unary function-symbol for each element of R. A structure with this signature is an R-module just in case the structure is an abelian group satisfying all identities

$$\begin{aligned} r(x + y) &= rx + ry, \\ (r + s)x &= rx + sx, \\ r(sx) &= rsx, \\ 1(x) &= x, \end{aligned}$$

where r and s are in R.

Example. Every Cartesian power of R is an R-module; in particular, R is an R-module.

A **submodule** of R is a substructure of R when R is considered as an R-module. Any subset A of R **generates** the submodule

$$(A),$$

which is the smallest submodule including A. A proper submodule of R is an **ideal** of R (although R is sometimes called an **improper** ideal of itself). If I is an ideal of R, then the **quotient** R/I is a ring, whose elements are the **cosets** $r + I$, where $r \in R$. (Here $r + I = \{r + a : a \in I\}$.)

Example. Any two integers a and b have a **greatest common divisor**, sometimes denoted (a, b), which can be found by the Euclidean algorithm; this integer generates the submodule of $\mathbf{Z}$ that is also denoted (a, b). Thus every ideal of $\mathbf{Z}$ is **principal**—generated by a single element. If n is a non-zero integer, then the quotient $Z/(n)$ is finite, and its universe can be identified with n. The quotient $\mathbf{Z}/(0)$ is $\mathbf{Z}$ itself.

If $h : R \to S$ is a homomorphism of rings, then its **kernel** comprises a in R such that $h(a) = 0$; this kernel is an ideal of R. Every ideal I of R is the kernel of the quotient-map from R to R/I.

Example. Suppose $\mathbf{a} \in \mathbf{C}^m$. Then there is a ring-homomorphism from $\mathbf{C}[x_0, \dots, x_{n+m-1}]$ to $\mathbf{C}[x_0, \dots, x_{n-1}]$, namely

$$t(x_0, \dots, x_{n+m-1}) \longmapsto t(x_0, \dots, x_{n-1}, \mathbf{a}).$$

The kernel is an ideal.

An ideal I of R is **prime** if the complement $R - I$ is closed under multiplication. An ideal of R is **maximal** if no ideal of R properly includes it.

Theorem. *Suppose I is an ideal of the commutative ring R. Then:*

- *I is prime if and only if R/I is an integral domain;*
- *I is maximal if and only if R/I is a field.*

A corollary of the theorem is that maximal ideals are prime.

Examples. The prime ideals of $\mathbf{Z}$ are the ideals (p), where p is a prime number; these ideals are maximal. Hence the quotients $\mathbf{Z}/(p)$ are fields, which can be denoted $\mathbf{F}_p$. The quotient $\mathbf{C}[x]/(x^2)$ is not an integral domain, since (x^2) is not prime. The quotient $\mathbf{C}[x]/(x)$ is just $\mathbf{C}$, so (x) is a maximal ideal.

7 Boolean algebras

An essential and notationally exceptional example is the *Boolean algebra* of subsets of a set Ω; this structure is the tuple

$$(\mathcal{P}(\Omega), \cap, \cup, {}^c, \emptyset, \Omega),$$

but we shall consider the signature of Boolean algebras to be the set

$$\{\wedge, \vee, \neg, 0, 1\}.$$

A **Boolean ring** is a ring satisfying

$$x^2 = x.$$

In particular, such a ring satisfies $(x + y)^2 = x + y$, hence

$$xy + yx = 0;$$

replacing y with x, we get $2x = 0$, hence

$$-x = x;$$

so the signature of Boolean rings can be considered to be $\{+, \cdot, 0, 1\}$. We also get $xy = yx$, so the ring is commutative. We have $x(1+x) = 0$, so if x is a unit, then $1 + x = 0$, so $x = 1$. Thus also every nonzero nonunit of a Boolean ring is a zero-divisor. Hence the only Boolean integral domain is the two-element ring $\{0, 1\}$, which is the field $\mathbf{F}_2$. Therefore prime ideals of Boolean rings are maximal, since the quotient of a Boolean ring by an ideal is Boolean.

In terms in the signature of Boolean algebras, customarily **negation** $(\neg)$ has priority over **conjunction** $(\wedge)$ and **disjunction** $(\vee)$. A structure in this signature *is* a **Boolean algebra** if it can be expanded to a signature containing $+$ in such a way that:

- the identities

$$x \vee y = x + y + (x \wedge y),$$
$$\neg x = x + 1$$

  are satisfied, and

- this expansion, reduced to the signature $\{+, \wedge, 0, 1\}$, is a Boolean ring.

If such an expansion is possible, then it is obtained by defining

$$x + y = (x \wedge \neg y) \vee (y \wedge \neg x).$$

The algebra $(\mathcal{P}(\Omega), \cap, \cup, {}^c, \emptyset, \Omega)$ is a Boolean algebra, since the required expansion is obtained by interpreting $+$ as **symmetric difference**, $\triangle$.

Any Boolean algebra has a partial order $\leqslant$ such that

$$x \leqslant y \iff x \wedge y = x;$$

its interpretation in $\mathcal{P}(\Omega)$ is *inclusion* $(\subseteq)$.

An **ideal** of a Boolean algebra is just an ideal of the corresponding ring. A **filter** of a Boolean algebra is *dual* to an ideal, so F is a filter just in case $\{\neg x : x \in F\}$ is an ideal. An **ultrafilter** is dual to a maximal ideal. So, F is a filter just in case

$$\begin{aligned}
&1 \in F, \\
&x, y \in F \implies x \wedge y \in F, \\
&x \in F \text{ and } x \leqslant y \implies y \in F, \\
&0 \notin F;
\end{aligned}$$

also, a filter F is an ultrafilter just in case

$$x \vee y \in F \implies x \in F \text{ or } y \in F,$$

equivalently, $x \notin F \implies \neg x \in F$.

The set of ultrafilters of a Boolean algebra is the **Stone-space** of the algebra. For every element x of a Boolean algebra, the corresponding Stone-space has a subset $[x]$ comprising the ultrafilters containing x. Then

$$[x] \cap [y] = [x \wedge y]$$

since the elements of these sets are filters; since they are ultrafilters, we have also

$$\begin{aligned}
[x] \cup [y] &= [x \vee y], \\
[x]^c &= [\neg x].
\end{aligned}$$

Finally, $[1]$ is the whole Stone-space, and $[0]$ is empty. Therefore the map

$$x \longmapsto [x]$$

is a homomorphism of Boolean algebras; it is an embedding, since $[x]$ is non-empty when $x \neq 0$.

A **lower bound** of a subset A of a Boolean algebra is an element a of the algebra such that

$$a \leqslant x$$

whenever $x \in A$; this lower bound is an **infimum** of A if $b \leqslant a$ whenever b is a lower bound of A. Infima are unique when they exist; but they may not exist. However,

$$\inf\{x, y\} = x \wedge y,$$

so infima of finite sets exist. Also, if $A \subseteq \mathcal{P}(\Omega)$, then $\inf A$ is the *intersection* of A. Thus every Boolean algebra embeds in an algebra where infima exist. However, an embedding need not preserve infima.

Example. Let A comprise the *cofinite* subsets of ω. Then $\inf A = \emptyset$. However, A is a filter of $\mathcal{P}(\omega)$, so A is included in an ultrafilter F. In the Stone-space,

$$F \in [x]$$

whenever $x \in A$; so $[\emptyset]$ is not the infimum of $\{[x] : x \in A\}$.

A **topology** for a set Ω is a substructure of $(\mathcal{P}(\Omega), \cap, \cup, 0, 1)$ that is closed under *arbitrary* intersection. (So the topology contains, for each of its subsets, the infimum that exists in $\mathcal{P}(\Omega)$.) The elements of the topology are the **closed** sets; their complements are **open**. A **basis** for a topology is just a substructure of $(\mathcal{P}(\Omega), \cup, 0, 1)$; the closed sets are then intersections of sets in the basis.

A topology is **Hausdorff** if any two distinct elements of the underlying set are contained in disjoint open sets.

A subset of $\mathcal{P}(\Omega)$ has the **finite-intersection property** if it generates a (proper) filter. A topology for Ω is **compact** if every collection of closed sets with the finite-intersection property has non-empty intersection. It is enough that these closed sets be in the basis, if there is one.

In particular, the subsets $[x]$ of a Stone-space compose a basis for a topology, and these basic sets are clopen. The topology is Hausdorff, since two distinct points of the space are respectively contained in some disjoint sets $[x]$ and $[\neg x]$.

Suppose B is a subset of a Boolean algebra. Then the following are equivalent:

- the collection $\{[x] : x \in B\}$ has the finite-intersection property;
- the set B generates a filter of the algebra;
- B included in an ultrafilter of this algebra;
- $\{[x] : x \in B\}$ has nonempty intersection.

Thus the topology of the Stone-space is compact. Consequently, every clopen set is one of the sets $[x]$.

Of the nonempty set Ω, we can see the Boolean ring $\mathcal{P}(\Omega)$ of its subsets as a compact **topological ring**. For, we can identify any subset A of Ω with its **characteristic function**, the map from Ω to $\mathbf{F}_2$ taking x to 1 just in case $x \in A$. The set of such maps can be denoted $\mathbf{F}_2^{\Omega}$. With the **discrete** topology, in which every subset is closed, $\mathbf{F}_2$ is a compact topological ring. Therefore on $\mathbf{F}_2^{\Omega}$ is induced a ring-structure and a compatible topology—the **product-**topology or topology of **pointwise convergence**, compact in this case since $\mathbf{F}_2$ is compact. The induced ring-structure makes the bijection from $\mathcal{P}(\Omega)$ to $\mathbf{F}_2^{\Omega}$ a homomorphism. In the induced topology, every finite subset of Ω determines for the zero-map on Ω an open neighborhood, comprising those maps into $\mathbf{F}_2$ that are zero on that finite subset. Translating such a neighborhood by an element of $\mathbf{F}_2^{\Omega}$ gives an open neighborhood of that element, and every open subset of $\mathbf{F}_2^{\Omega}$ is a union of such neighborhoods; the finite unions are precisely the clopen subsets.

8 Propositional logic

The terms in the signature of Boolean algebras—the **Boolean terms**—can be considered as strings of symbols generated by the following rules:

- each constant-symbol 0 or 1 is a term;
- each symbol x_i for a coordinate projection is a term;
- if t and u are terms, then so are $(t \wedge u)$ and $(t \vee u)$ and $\neg t$.

A term here is n-ary just in case $i < n$ whenever x_i appears in the term. Instead of $(\cdots(((t_0 * t_1) * t_2) * \cdots * t_{n-1})$ we can write

$$t_0 * t_1 * t_2 * \cdots * t_{n-1},$$

where each $*$ is (independently) $\wedge$ or $\vee$.

Lemma. *Every n-ary function on $\mathbf{F}_2$ is the interpretation of an n-ary Boolean term.*

Proof. Suppose f be an n-ary function on $\mathbf{F}_2$, and let $\mathbf{a}^0, \dots, \mathbf{a}^{m-1}$ be the elements of $\mathbf{F}_2^n$ at which f is 1. If $m = 0$, then f is the interpretation of 0. If $m > 0$, then f is the interpretation of

$$t^0 \vee \cdots \vee t^{m-1},$$

where t^j is $u_0^j \wedge \cdots \wedge u_{n-1}^j$, where u_i^j is x_i, if $a_i^j = 1$, and otherwise is $\neg x_i$. $\square$

The Boolean terms can be considered as the *propositional formulas* composing a *propositional logic*. The constant-symbols 0 and 1 can then be taken to stand for **false** and **true** statements, respectively; an element of $\mathbf{F}_2^\omega$ is a **truth-assignment** to the **propositional variables** x_i, and under such an assignment σ, a propositional formula t takes on the **truth-value**

$$t^{\mathbf{F}_2}(\sigma(0), \dots, \sigma(n-1))$$

if t is n-ary. Write $\langle \sigma, t \rangle$ for the truth-value of t under σ. A *model* for a set of propositional formulas is a truth-assignment σ sending the set to 1 under the map $t \mapsto \langle \sigma, t \rangle$.

Theorem (Compactness for sentential logic). *A set of propositional formulas has a model if each finite subset does.*

Proof. If a set of sentences t satisfies the hypothesis, then the collection of closed subsets $\{\sigma : \langle \sigma, t \rangle = 1\}$ of $\mathbf{F}_2^\omega$ has the finite-intersection property. $\square$

The sets $\{\sigma : \langle \sigma, t \rangle = 1\}$ are precisely the clopen subsets of $\mathbf{F}_2^\omega$.

9 Relations and formulas

From the relations $R^{\mathcal{M}}$ and the interpretations $t^{\mathcal{M}}$ of terms t, new relations on M can be derived by various techniques. These relations will be the 0-**definable** relations of $\mathcal{M}$, and each of them will be the interpretation of a **formula** of $\mathcal{L}$. (The **definable** relations of $\mathcal{M}$ are the interpretations of formulas of $\mathcal{L}(M)$.) Distinctions are made according to which techniques are needed to derive the relations.

The **atomic** formulas are given thus:

- If t and u are n-ary terms, then there is an n-ary atomic formula $t = u$ whose interpretation $(t = u)^{\mathcal{M}}$ is $\{\mathbf{a} \in M^n : t^{\mathcal{M}}(\mathbf{a}) = u^{\mathcal{M}}(\mathbf{a})\}$.
- If $t_0, \dots, t_{n-1}$ are m-ary terms, and R is n-ary, then there is an m-ary atomic formula—call it $R(t_0, \dots, t_{n-1})$—whose interpretation is $\{\mathbf{a} \in M^m : (t_0^{\mathcal{M}}(\mathbf{a}), \dots, t_{n-1}^{\mathcal{M}}(\mathbf{a})) \in R^{\mathcal{M}}\}$.

(In particular, $R(x_0, \dots, x_{n-1})^{\mathcal{M}} = R^{\mathcal{M}}$.)

A **literal** is an atomic formula or its **negation**. The negation of an atomic formula α can be written

$$\neg\alpha,$$

but the negation of $t = u$ is also $t \neq u$. The interpretation in $\mathcal{M}$ of $\neg\alpha$ is the complement of $\alpha^{\mathcal{M}}$.

A literal is an example of a *basic* or *quantifier-free* formula. If t is an n-ary Boolean term, and $\phi_0, \dots, \phi_{n-1}$ are m-ary atomic formulas, then there is an m-ary **basic** or **quantifier-free** formula, say $t(\phi_0, \dots, \phi_{n-1})$, whose interpretation in $\mathcal{M}$ is

$$t^{\mathcal{P}(M^m)}(\phi_0^{\mathcal{M}}, \dots, \phi_{n-1}^{\mathcal{M}}).$$

If we identify formulas that have indistinguishable interpretations in every structure, then the set of basic formulas is a Boolean algebra generated by the atomic formulas. (This assumes that the Boolean terms 0 and 1 are also n-ary formulas. If $n > 0$, then these are identified respectively with $x_0 \neq 0$ and $x_0 = x_0$. If $n = 0$, then the formulas might be written $\perp$ and $\top$; but some model-theorists don't use such formulas.)

The set of **formulas** is then the smallest Boolean algebra containing the atomic formulas and closed under the operation of **existential quantification**; this converts an $n+1$-ary formula ϕ into an n-ary formula $\exists x_n\, \phi$ whose interpretation is the image of $\phi^{\mathcal{M}}$ under the map

$$(a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1}) : M^{n+1} \to M^n.$$

The Boolean algebra of n-ary formulas of $\mathcal{L}$ can be denoted $\mathrm{Fm}^n(\mathcal{L})$.

The formula $\neg\exists x_n\, \phi$ is also denoted $\forall x_n\, \neg\phi$, and $\neg\phi \vee \psi$ is denoted $\phi \to \psi$.

Lemma. *If ϕ is an n-ary formula, and $t_0, \dots, t_{n-1}$ are m-ary terms, then there is an m-ary formula $\phi(t_0, \dots, t_{n-1})$ with the obvious interpretation.*

In particular, if it is not also $n-1$-ary, then an n-ary formula ϕ is the same as the formula $\phi(x_0, \dots, x_{n-1})$.

The A-definable relations of $\mathcal{M}$ are the interpretations in $\mathcal{M}$ of formulas of $\mathcal{L}(A)$. In particular, they are the sets $\phi(a_0, \dots, a_{m-1}, x_0, \dots, x_{n-1})^{\mathcal{M}}$, where ϕ is an $m+n$-ary formula of $\mathcal{L}$, and $\mathbf{a}$ is a tuple from A.

Sentences are 0-ary formulas.

10 Elementary embeddings

Suppose $\mathcal{M}$ and $\mathcal{N}$ are members of $\mathfrak{Mod}(\mathcal{L})$. We can now say that an embedding of $\mathcal{M}$ in $\mathcal{N}$ is a map $h : M \to N$ such that

$$h^{-1}(\phi^{\mathcal{N}}) = \phi^{\mathcal{M}}$$

for all basic formulas ϕ of $\mathcal{L}$ (or just all literals of $\mathcal{L}$); if the same holds for *all* formulas ϕ of $\mathcal{L}$, then h is an **elementary embedding**. If $\mathcal{M} \subseteq \mathcal{N}$, and the inclusion-map of M in N is an *elementary* embedding, we write

$$\mathcal{M} \preccurlyeq \mathcal{N}$$

and say $\mathcal{M}$ is an **elementary** substructure of $\mathcal{N}$.

Lemma (Tarski–Vaught). *Suppose $\mathcal{M} \subseteq \mathcal{N}$. Then $\mathcal{M} \preccurlyeq \mathcal{N}$, provided that*

$$\phi(\mathbf{a}, x_0)^{\mathcal{N}} \cap M$$

is nonempty whenever $\phi(\mathbf{a}, x_0)^{\mathcal{N}}$ is, for all $\mathcal{L}$-formulas ϕ and all tuples $\mathbf{a}$ from M.

Proof. Let Σ comprise the formulas ϕ such that

$$\phi(x_0, \dots, x_{n-1})^{\mathcal{M}} = \phi(x_0, \dots, x_{n-1})^{\mathcal{N}} \cap M^n. \qquad (*)$$

Then Σ contains all the basic formulas and is closed under the Boolean operations. Suppose ϕ is in Σ and $\mathbf{a}$ is in M^n. Then

$$\phi(\mathbf{a}, x_0)^{\mathcal{M}} = \phi(\mathbf{a}, x_0)^{\mathcal{N}} \cap M.$$

By hypothesis then, $\phi(\mathbf{a}, x_0)^{\mathcal{M}}$ and $\phi(\mathbf{a}, x_0)^{\mathcal{N}}$ are alike empty or not. Hence $(*)$ holds, *mutatis mutandis*, with $\exists x_{n-1}\phi$ in place of ϕ. Therefore $\Sigma = \mathrm{Fm}(\mathcal{L})$. $\square$

11 Models and theories

Suppose $\phi \in \mathrm{Fm}^n(\mathcal{L})$, and $\mathbf{a} \in M^n$, so that $\phi(\mathbf{a}) \in \mathrm{Fm}^0(\mathcal{L}(M))$. Then

$$\begin{aligned}\phi(\mathbf{a})^{\mathcal{M}} &= \{() \in M^0 : (a_0^{\mathcal{M}}(), \dots, a_{n-1}^{\mathcal{M}}()) \in \phi^{\mathcal{M}}\} \\ &= \{() \in M^0 : \mathbf{a} \in \phi^{\mathcal{M}}\}.\end{aligned}$$

So $\phi(\mathbf{a})^{\mathcal{M}} = 1$ if $\mathbf{a} \in \phi^{\mathcal{M}}$, and in this case we write

$$\mathcal{M} \models \phi(\mathbf{a});$$

if $\mathbf{a} \in M^n - \phi^{\mathcal{M}}$, then $\phi(\mathbf{a})^{\mathcal{M}} = 0$, and $\mathcal{M} \models \neg\phi(\mathbf{a})$. The map $h : M \to N$ is an elementary embedding just in case

$$\mathcal{M} \models \phi(\mathbf{a}) \iff \mathcal{N} \models \phi(h(\mathbf{a}))$$

for all such ϕ and $\mathbf{a}$.

If $\mathcal{K}$ is a subclass of $\mathfrak{Mod}(\mathcal{L})$, then the **theory** $\mathrm{Th}(\mathcal{K})$ of $\mathcal{K}$ is the subset of $\mathrm{Fm}^0(\mathcal{L})$ comprising σ such that $\mathcal{M} \models \sigma$ whenever $\mathcal{M} \in \mathcal{K}$; this subset is a filter, if $\mathcal{K}$ is nonempty; otherwise it contains every sentence. In general, a **theory** of $\mathcal{L}$ is $\mathrm{Fm}^0(\mathcal{L})$ or a filter of it; a **consistent** theory is a proper filter; a **complete** theory is an ultrafilter. A **model** of a set Σ of sentences is a structure $\mathcal{M}$ such that $\Sigma \subseteq \mathrm{Th}(\mathcal{M})$. We write

$$\Sigma \models \sigma$$

if every model of Σ is a model of σ (that is, of $\{\sigma\}$). We write

$$\Sigma \vdash \sigma$$

if σ is in the theory generated by Σ. If $\Sigma \vdash \sigma$, then $\Sigma \models \sigma$.

12 Compactness

It is a consequence of the following that $\Sigma \vdash \sigma$ if $\Sigma \models \sigma$.

Theorem (Compactness). *Every consistent theory has a model.*

Proof. Let T be a consistent theory in the signature $\mathcal{L}$. We shall extend $\mathcal{L}$ to a signature $\mathcal{L}'$, and extend T to a complete theory T' of $\mathcal{L}'$. We shall do this in such a way that, for every unary formula ϕ of $\mathcal{L}'$, there will be a constant-symbol c_ϕ not appearing in ϕ such that

$$T' \vdash \exists x_0\, \phi \to \phi(c_\phi).$$

Then T' and the constant-symbols c_ϕ will determine a structure $\mathcal{M}$ in the following way. The universe of $\mathcal{M}$ will consist of equivalence-classes $[c_\phi]$ of the symbols c_ϕ, where

$$[c_\phi] = [c_\psi] \iff T' \vdash c_\phi = c_\psi.$$

Then we require

$$\phi^{\mathcal{M}} = \{[\mathbf{c}] : T' \vdash \phi(\mathbf{c})\} \tag{$*$}$$

for all basic formulas ϕ of $\mathcal{L}'$ and all tuples $\mathbf{c}$ of symbols c_ϕ. The requirements $(*)$ do make sense. In particular, $c_\phi^{\mathcal{M}} = [c_\phi]$. The requirements determine a well-defined structure, since T' is complete.

If T' is as claimed, then $(*)$ holds for all formulas ϕ; we show this by induction. If ϕ is an n-ary formula, and $[\mathbf{c}]$ is an $(n-1)$-tuple from M, let d be the constant-symbol determined by the unary formula $\phi(\mathbf{c}, x_0)$. If $(*)$ holds for ϕ, then we have:

$$\begin{aligned}
[\mathbf{c}] \in (\exists x_n\, \phi)^{\mathcal{M}} &\implies \mathcal{M} \models \phi(\mathbf{c}, [e]),\ \text{some } [e] \text{ in } M \\
&\implies T' \vdash \phi(\mathbf{c}, e) \\
&\implies T' \vdash \exists x_0\, \phi(\mathbf{c}, x_0) \\
&\implies T' \vdash \phi(\mathbf{c}, d) \\
&\implies \mathcal{M} \models (\mathbf{c}, [d]) \\
&\implies [\mathbf{c}] \in (\exists x_n\, \phi)^{\mathcal{M}};
\end{aligned}$$

so $(*)$ holds with $\exists x_n\, \phi$ in place of ϕ.

Once $(*)$ holds for all formulas ϕ, then in particular it holds when ϕ is a sentence in T; so $\mathcal{M} \models T$.

It remains to find T' as desired. First we construct a chain $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots$ of signatures, where $\mathcal{L}_{n+1} - \mathcal{L}_n$ consists of a constant-symbol c_ϕ for each unary formula ϕ in $\mathcal{L}_n$. Taking the union of the chain gives $\mathcal{L}'$.

Now we work in the Stone space of $\mathrm{Fm}^0(\mathcal{L}')$. We claim that the collection

$$\{[\sigma] : \sigma \in T\} \cup \{[\forall x_0\, \neg\phi \vee \phi(c_\phi)] : \phi \in \mathrm{Fm}^1(\mathcal{L}')\}$$

of closed sets has the finite-intersection property; from this, by compactness, we can take T' to be an element of the intersection.

To establish the f.i.p., suppose that $[\psi]$ is a nonempty finite intersection of sets in the collection. Then $\psi \in \mathrm{Fm}^0(\mathcal{L}_n)$ for some n. If $\phi \in \mathrm{Fm}^1(\mathcal{L}') - \mathrm{Fm}^1(\mathcal{L}_{n-1})$, then c_ϕ does not appear in ψ. If also $[\psi] \cap [\forall x_0\, \neg\phi]$ is empty, then

$$[\psi] \cap [\phi(c_\phi)]$$

is nonempty; for, if $\mathcal{M} \models \psi \wedge \exists x_0\, \phi$, then we may assume $\mathcal{M} \models \psi \wedge \phi(c_\phi)$. □

Theorem. *Suppose $\mathcal{N} \in \mathfrak{Mod}(\mathcal{L})$, and κ is a cardinal such that*

$$\aleph_0 + |\mathcal{L}| \leqslant \kappa \leqslant |N|.$$

Then there is $\mathcal{M}$ in $\mathfrak{Mod}(\mathcal{L})$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$ and $|M| = \kappa$.

Proof. Use the proof of Compactness, with $\mathrm{Th}(\mathcal{N})$ for T. We can choose T', and we can choose $c_\phi^{\mathcal{N}}$ in N, so that $\mathcal{N} \models T'$. Then we may assume $M \subseteq N$, and so $\mathcal{M} \preccurlyeq \mathcal{N}$ by the Tarski–Vaught test. By construction, $|M| \leqslant |\mathcal{L}'| = \aleph_0 + |\mathcal{L}|$.

To ensure $M = \kappa$, we first add κ-many new constant-symbols to $\mathcal{L}$ and let their interpretations in $\mathcal{N}$ be distinct. □

Example. In the signature $\{\in\}$ of set-theory, any infinite structure has a countably infinite elementary substructure, even though the power-set of an infinite set is uncountable.

Corollary. *Suppose $\mathcal{A}$ is an infinite $\mathcal{L}$-structure and $|A| + |\mathcal{L}| \leqslant \kappa$. Then there is $\mathcal{M}$ in $\mathfrak{Mod}(\mathcal{L})$ such that $\mathcal{A} \preccurlyeq \mathcal{M}$ and $|M| = \kappa$.*

Proof. Let $\{c_\mu : \mu < \kappa\}$ be a set of new constant-symbols, and let T be the theory generated by $\mathrm{Th}(\mathcal{A}_A)$ and $\{c_\mu \neq c_\nu : \mu \neq \nu\}$. Use Compactness to get a model $\mathcal{N}$ of T; then use the last Theorem to get $\mathcal{M}$ as desired. □

References

[1] C. C. Chang and H. J. Keisler. *Model theory.* North-Holland Publishing Co., Amsterdam, third edition, 1990.

[2] Wilfrid Hodges. *Model Theory.* Cambridge University Press, 1993.

[3] Bruno Poizat. *A course in model theory.* Springer-Verlag, New York, 2000. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author.