NUMBER-THEORY EXERCISES, II.III

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Exercise 1. Verify that the integers of a quadratic field do compose a ring.

Exercise 2. Suppose $\tau = (15 + 3\sqrt{17})/4$. Find A, B, and C in $\mathbb{Z}$ such that $A\tau^2 + B\tau + C = 0$ and $\gcd(A, B, C) = 1$.

Exercise 3. Suppose $A\tau^2 + B\tau + C = 0$ for some A, B, and C in $\mathbb{Z}$, where $A > 0$ and $\gcd(A, B, C) = 1$.

(a) Show $\langle 1, A\bar{\tau}\rangle\langle 1, \tau\rangle = \langle 1, \tau\rangle$.
(b) Show $\langle A, A\bar{\tau}\rangle\langle 1, \tau\rangle = \langle 1, A\bar{\tau}\rangle$.
(c) Using (a) and (b), show $\mathfrak{O}_\Lambda = \langle 1, A\bar{\tau}\rangle$, where $\Lambda = \langle 1, \tau\rangle$.

Exercise 4. Let Λ be the lattice

$$\left\langle \frac{3 + 5\sqrt{6}}{2}, \frac{6 + \sqrt{6}}{3} \right\rangle$$

of $\mathbb{Q}(\sqrt{6})$. Find $\mathfrak{O}_\Lambda$.

Exercise 5. Suppose $\tau \in \mathbb{C} \smallsetminus \mathbb{Q}$. Show that the following are equivalent:

(i) $A\tau^2 + B\tau + C = 0$ for some A, B, and C in $\mathbb{Z}$;
(ii) $\alpha\langle 1, \tau\rangle \subseteq \langle 1, \tau\rangle$ for some α in $\mathbb{C} \smallsetminus \mathbb{Z}$.

Exercise 6. Let $f(x, y)$ be the quadratic form

$$60x^2 + 224xy - 735y^2.$$

(a) Find the discriminant of f in the form $n\sqrt{d}$, where n and d are rational integers, and d is square-free.
(b) Find all solutions from $\mathbb{Z}$ of $f(x, y) = 1$.
(c) Find all solutions from $\mathbb{Z}$ of $f(x, y) = 6$.

Exercise 7. For every lattice Λ of a quadratic field K, show that the units of $\mathfrak{O}_\Lambda$ are just the units of $\mathfrak{O}_K$ that are in $\mathfrak{O}_\Lambda$.

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