NUMBER-THEORY EXERCISES, II.II

DAVID PIERCE

Exercise 1. If d is a positive non-square rational integer, prove $\sqrt{d}$ is irrational.

Exercise 2. Find a greatest common divisor α of the Gaussian integers $27 + 55\mathrm{i}$ and $20 + 18\mathrm{i}$, and solve

$$(27 + 55\mathrm{i})\xi + (20 + 18\mathrm{i})\eta = \alpha.$$

Exercise 3. Find all solutions of the Diophantine equation

$$x^2 + y^2 = 1170.$$

Exercise 4. Assuming n is positive, prove that the number of solutions of the Diophantine equation

$$x^2 + y^2 = n$$

is 4 times the excess of the number of positive factors of n that are congruent to 1 *modulo* 4 over the number that are congruent to 3 *modulo* 4.

Exercise 5.

(a) Characterize (by describing their prime factorizations) those Gaussian integers α such that $|\alpha|^2$ is square as a rational integer.
(b) Use this characterization to solve the Diophantine equation

$$x^2 + y^2 = z^2.$$

Exercise 6. The polynomial $x^2 + x + 1$ has two conjugate roots. Let ω be the root with positive imaginary part.

(a) Write ω in radicals.
(b) Sketch $\mathbb{Z}[\omega]$ as a subset of the complex plane.
(c) Letting $\mathrm{N}(z) = |z|^2$, show that $\mathrm{N}(\alpha) \in \mathbb{N}$ when $\alpha \in \mathbb{Z}[\omega]$.
(d) Express $\mathrm{N}(x + \omega y)$ in terms of x and y.
(e) Show that $\mathbb{Z}[\omega]$ with $z \mapsto \mathrm{N}(z)$ is a Euclidean domain.

(The elements of $\mathbb{Z}[\omega]$ are the **Eisenstein integers.**)

Mathematics Dept, Middle East Tech. Univ., Ankara 06531, Turkey
E-mail address: dpierce@metu.edu.tr
URL: http://www.math.metu.edu.tr/~dpierce/courses/366/

Date: March 11, 2008.