ELEMENTARY NUMBER THEORY II, EXAMINATION III SOLUTIONS

Problem 1. Suppose $\sqrt{2} = [a_0; a_1, a_2, \dots]$, and as usual let $p_n/q_n = [a_0; a_1, \dots, a_n]$. Find rational integers a, b, k, and ℓ such that

$$p_n + q_n\sqrt{2} = (a + b\sqrt{2})(k + \ell\sqrt{2})^n$$

for all positive rational integers n.

Solution. First compute the expansion of $\sqrt{2}$:

$$\begin{aligned} & & a_0 &= 1, & \xi_0 &= \sqrt{2} - 1; \\ \frac{1}{\sqrt{2}-1} &= \sqrt{2} + 1, & a_1 &= 2, & \xi_1 &= \sqrt{2} - 1 = \xi_0. \end{aligned}$$

So $\sqrt{2} = [1, \overline{2}]$. In particular, the period has length 1, so

$$p_{n+1} + q_{n+1}\sqrt{2} = (p_n + q_n\sqrt{2})(p_0 + q_0\sqrt{2}).$$

Since $p_0/q_0 = 1/1$, we conclude

$$p_n + q_n\sqrt{2} = (1 + \sqrt{2})(1 + \sqrt{2})^n.$$

(This is justified by Theorem 20 of the notes. Alternatively, one may note that

$$\frac{p_{n+1}}{q_{n+1}} = \left[1, 1 + \frac{p_n}{q_n}\right] = 1 + \frac{q_n}{p_n + q_n} = \frac{p_n + 2q_n}{p_n + q_n},$$

and both fractions are irreducible, so $(p_n + q_n\sqrt{2})(1 + \sqrt{2}) = p_n + 2q_n + (p_n + q_n)\sqrt{2} = p_{n+1} + q_{n+1}\sqrt{2}$.)

Problem 2. Here Λ and M are lattices in some quadratic field.

(a) Find $|\Lambda/M|$, that is, $(\Lambda : M)$, when
   (i) $\Lambda = \langle \alpha, \beta \rangle$, $M = \langle 2\alpha, 3\beta \rangle$;
   (ii) $\Lambda = \langle \alpha, \beta \rangle$, $M = \langle 2\alpha, \alpha + 3\beta \rangle$.

(b) Assuming $M \subseteq \Lambda$, find a number n such that $n\Lambda \subseteq M$.

Solution.

(a)
   (i) 6
   (ii) 6

(b) Let $n = (\Lambda : M)$. Indeed, we can write Λ as $\langle \alpha, \beta \rangle$, and then $M = \langle c\alpha, f\alpha + e\beta \rangle$ for some positive rational integers c, e, and f. Then $(\Lambda : M) = ce$, and $ce\Lambda = \langle ce\alpha, ce\beta \rangle \subseteq M$ since $ce\beta = c(f\alpha + e\beta) - f(c\alpha)$.

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Problem 3. In some quadratic field, find a lattice Λ such that $\mathrm{N}(\Lambda) = 1$, but $\Lambda \neq \mathfrak{O}_\Lambda$.

Solution. One strategy is to find a lattice $\langle \alpha, \beta \rangle$ whose norm is k^2 for some k; then Λ can be $\langle \alpha/k, \beta/k \rangle$. Assuming the quadratic field K is $\mathbb{Q}(\sqrt{d})$, where $d \equiv 2$ or $3 \pmod 4$, we can try letting $\langle \alpha, \beta \rangle = \langle k^2, \ell + \sqrt{d} \rangle$, where ℓ will be chosen so that the order is $\langle 1, \sqrt{d} \rangle$, that is, $\mathfrak{O}_K$. To compute this order, we have

$$\begin{aligned} x = \frac{\ell + \sqrt{d}}{k^2} &\implies k^2 x - \ell = \sqrt{d} \\ &\implies k^4 x^2 - 2k^2 \ell x + \ell^2 - d = 0 \\ &\implies k^2 x^2 - 2\ell x + \frac{\ell^2 - d}{k^2} = 0. \end{aligned}$$

It is enough now if $\gcd(k, 2\ell) = 1$, while $k^2 \mid \ell^2 - d$. We can achieve this by letting $k = 3$, $\ell = 5$, and $d = -2$. So

$$\Lambda = \left\langle 3, \frac{5 + \sqrt{-2}}{3} \right\rangle$$

is one possibility.

Problem 4. Letting $K = \mathbb{Q}(\sqrt{5})$ and $\mathfrak{O} = \mathfrak{O}_K$, for each p in $\{2, 3, 5, 7, 11\}$, find the prime factorization of $p\mathfrak{O}$ in $\mathfrak{O}$.

Solution. In the notation of our last theorem, $\Delta = d = 5$. Then 5 ramifies in $\mathfrak{O}$, and

$$5\mathfrak{O} = \left\langle 5, \frac{5 + \sqrt{5}}{2} \right\rangle^2.$$

Now we check solubility of $5 \equiv x^2 \pmod{4p}$ for the remaining p. There is no solution when $p \in \{2, 3, 7\}$. Indeed, when $p = 2$, just check the possibilities: $(\pm 1)^2 \equiv 1$; $(\pm 2)^2 \equiv 4$; $(\pm 3)^2 \equiv 1$; $4^2 \equiv 0$. In the other cases, we can show $5 \equiv x^2 \pmod p$ is insoluble by Legendre symbols and quadratic reciprocity: $(5/3) = (2/3) = -1$; $(5/7) = (7/5) = (2/5) = -1$. So 2, 3, and 7 are inert in $\mathfrak{O}$.

Finally, $(5/11) = (11/5) = (1/5) = 1$, and indeed $5 \equiv 7^2 \pmod{44}$. Then

$$11\mathfrak{O} = \left\langle 11, \frac{7 + \sqrt{5}}{2} \right\rangle \left\langle 11, \frac{7 - \sqrt{5}}{2} \right\rangle.$$

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