ELEMENTARY NUMBER THEORY II, EXAMINATION II

Instructions. Solve four of these five problems in 90 minutes. *İyi çalışmalar.*

Problem 1. Assuming $a > 0$, prove
$$\sqrt{a^2+1} = [a; \overline{2a}].$$

Problem 2. Let $K = \mathbb{Q}(\sqrt{5})$ and $\Lambda = \langle 1, \sqrt{5} \rangle$.
(a) Find the order $\mathfrak{O}_\Lambda$ (that is, $\{\xi \in K \colon \xi\Lambda \subseteq \Lambda\}$).
(b) Find the elements of $\mathfrak{O}_\Lambda$ having norm 1.

Problem 3. Solve in $\mathbb{Z}$:
$$x^2 + 2xy + 4y^2 = 19.$$

Problem 4.
(a) Prove that, for each n in $\mathbb{Z}$, there are a_n and b_n in $\mathbb{Z}$ such that
$$a_n + b_n\sqrt{21} = 2\Big(\frac{5+\sqrt{21}}{2}\Big)^n.$$
(b) Find a quadratic form $f(x, y)$ and a rational integer m such that each $(\pm a_n, \pm b_n)$ is a solution of
$$f(x, y) = m. \qquad (*)$$
(c) Prove that each solution of $(*)$ is $(\pm a_n, \pm b_n)$ for some n.

Problem 5.
(a) Find a quadratic field K, a lattice $\langle \alpha, \beta \rangle$ or Λ of K, and m in $\mathbb{Z}$ for which the function
$$(x, y) \mapsto x\alpha + y\beta$$
is a bijection between the solution-set (in $\mathbb{Z} \times \mathbb{Z}$) of
$$2x^2 - 3y^2 = 2 \qquad (\dagger)$$
and the solution-set in Λ of $\mathrm{N}(\xi) = m$.
(b) Describe a parallelogram Π in the plane $\mathbb{R}^2$ such that, for every solution (a, b) of $(\dagger)$, there is a solution (c, d) in Π such that
$$\frac{a\alpha + b\beta}{c\alpha + d\beta} \in \mathfrak{O}_\Lambda. \qquad (\ddagger)$$
(c) Find Π as in (b) with the additional condition that, if (a, b) and (c, d) are distinct solutions to $(\dagger)$ in Π, then $(\ddagger)$ fails.

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