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METU MATH 365, EXAM 1

Monday, November 8, 2010, at 13:40
Instructor: David Pierce

Instructions: Write your full name, and your student number, at the top of this page. Please work carefully. There are 6 numbered problems on 4 pages. It should be obvious to the grader how to read your solutions.

Problem 1. Let $\omega = \{0, 1, 2, \dots\}$. All variables in this problem range over ω. Given a and b such that $a \neq 0$, we define

$$\operatorname{rem}(b, a) = r,$$

if $b = ax + r$ for some x, and $r < a$.

a. Prove $\operatorname{rem}(a + b, n) = \operatorname{rem}(\operatorname{rem}(a, n) + \operatorname{rem}(b, n), n)$.

b. Prove $\operatorname{rem}(ab, n) = \operatorname{rem}(\operatorname{rem}(a, n) \cdot \operatorname{rem}(b, n), n)$.

Problem 2. Find integers k and ℓ, both greater than 1, such that, for all positive integers n,

$$k \mid 1965^{10n} + \ell.$$

Problem 3. Find two positive integers a and b such that, for all integers m and n, the integer $am - bn$ is a solution of the congruences

$$x \equiv m \pmod{999}, \qquad x \equiv n \pmod{1001}.$$

Problem 4. Letting $n = \sum_{j=1}^{408} j$, find an integer k such that $0 \leqslant k < 409$ and

$$408! \equiv k \pmod{n}.$$

Problem 5. With justification, find an integer n, greater than 1, such that, for all integers a,

$$a^n \equiv a \pmod{1155}.$$

Problem 6. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Suppose all we know about this set is:

(i) proofs by induction are possible;

(ii) addition can be defined on $\mathbb{N}$, and it satisfies

$$x + y = y + x, \qquad x + (y + z) = (x + y) + z;$$

(iii) multiplication can be defined by

$$x \cdot 1 = x, \qquad x \cdot (y + 1) = x \cdot y + x.$$

Prove

$$x \cdot y = y \cdot x.$$