NUMBER-THEORY EXERCISES, VII

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Exercise 1. $f(568) = f(638)$ when $f \in \{\tau, \sigma, \phi\}$.

Exercise 2. Solve:

(a) $n = 2\phi(n)$.
(b) $\phi(n) = \phi(2n)$.
(c) $\phi(n) = 12$. (Do this without a table. There are 6 solutions.)

Exercise 3. Find a sequence $(a_n : n \in \mathbb{N})$ of positive integers such that

$$\lim_{n\to\infty} \frac{\phi(a_n)}{a_n} = 0.$$

(If you assume that there *is* an answer to this problem, then it is not hard to see what the answer must be. To actually *prove* that the answer is correct, recall that, formally,

$$\sum_n \frac{1}{n} = \prod_p \frac{1}{1 - \frac{1}{p}},$$

so $\lim_{n\to\infty} \prod_{k=0}^{n} \frac{1}{1 - \frac{1}{p_k}} = \infty$ if $(p_k : k \in \mathbb{N})$ is the list of primes.)

Exercise 4. (a) Show $a^{100} \equiv 1 \pmod{1000}$ if $\gcd(a, 1000) = 1$.
(b) Find n such that $n^{101} \not\equiv n \pmod{1000}$.

Exercise 5. (a) Show $a^{24} \equiv 1 \pmod{35}$ if $\gcd(a, 35) = 1$.
(b) Show $a^{13} \equiv a \pmod{35}$ for all a.
(c) Is there n such that $n^{25} \not\equiv n \pmod{35}$?

Exercise 6. If $\gcd(m, n) = 1$, show $m^{\phi(n)} \equiv n^{\phi(m)} \pmod{mn}$.

Exercise 7. If n is odd, and is not a prime power, and if $\gcd(a, n) = 1$, show $a^{\phi(n)/2} \equiv 1 \pmod{n}$.

Exercise 8. Solve $5^{10000}x \equiv 1 \pmod{153}$.

Exercise 9. Prove $\sum_{d|n} \mu(d)\phi(d) = \prod_{p|n}(2 - p)$.

Exercise 10. If n is **squarefree** (has no factor p^2), and $k \in \mathbb{N}$, show

$$\sum_{d|n} \sigma(d^k)\phi(d) = n^{k+1}.$$

Exercise 11. $\sum_{d|n} \sigma(d)\phi\Big(\frac{n}{d}\Big) = n\tau(n)$.

Exercise 12. $\sum_{d|n} \tau(d)\phi\Big(\frac{n}{d}\Big) = \sigma(n)$.

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