NUMBER-THEORY EXERCISES, VI

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The variables n, k, and d range over the positive integers.

Exercise 1. Assuming p is an *odd* prime:

(a) $(p-1)! \equiv p-1 \pmod{1+2+\cdots+(p-1)}$;
(b) $1\cdot 3\cdots(p-2) \equiv (-1)^{(p-1)/2}\cdot(p-1)\cdot(p-3)\cdots 2 \pmod{p}$;
(c) $1\cdot 3\cdots(p-2) \equiv (-1)^{(p-1)/2}\cdot 2\cdot 4\cdots(p-1) \pmod{p}$;
(d) $1^2\cdot 3^2\cdots(p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$.

Exercise 2. $\tau(n) \leqslant 2\sqrt{n}$.

Exercise 3. $\tau(n)$ is odd if and only if n is square.

Exercise 4. Assuming n is odd: $\sigma(n)$ is odd if and only if n is square.

Exercise 5. $\displaystyle\sum_{d\mid n}\frac{1}{d} = \frac{\sigma(n)}{n}$.

Exercise 6. $\{n\colon \tau(n)=k\}$ is infinite (when $k>1$), but $\{n\colon \sigma(n)=k\}$ is finite.

Exercise 7. Let $m\in\mathbb{Z}$. The number-theoretic function $n\mapsto n^m$ is multiplicative.

Exercise 8. Let $\omega(n)$ be the number of *distinct* prime divisors of n, and let m be a non-zero integer. Then $n\mapsto m^{\omega(n)}$ is multiplicative.

Exercise 9. Let $\Lambda(n) = \begin{cases}\log p, & \text{if } n=p^m \text{ for some positive } m;\\ 0, & \text{otherwise.}\end{cases}$

(a) $\log n = \displaystyle\sum_{d\mid n}\Lambda(d)$.
(b) $\Lambda(n) = \displaystyle\sum_{d\mid n}\mu\left(\frac{n}{d}\right)\log d$.
(c) $\Lambda(n) = -\displaystyle\sum_{d\mid n}\mu(d)\log d$.

Exercise 10. Suppose $n = {p_1}^{k(1)}\cdots{p_r}^{k(r)}$, where the p_i are distinct.

(a) If f is multiplicative and non-zero, then $\displaystyle\sum_{d\mid n}\mu(d)\cdot f(d) = \prod_{i=1}^{r}(1-f(p_i))$;
(b) $\displaystyle\sum_{d\mid n}\mu(d)\cdot\tau(d) = (-1)^r$.

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