Second Examination *solutions*

Math 320, David Pierce

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Problem 1. *Write the following ordinals in Cantor normal form (that is, in base* ω*). All exponents should themselves be in normal form. (*Exception: *The exponent* 1 *and the coefficient* 1 *need not be written. In strict normal form,* ω *should be written as* ω^{ω^0} *or even* $\omega^{\omega^{0\cdot 1}}\cdot 1$*; but this is not required.)*

Solution.

a) $\omega+2$ [already in normal form]

b) $2+\omega=\omega$

c) $\omega^2+\omega$ [already in normal form]

d) $\omega+\omega^2=\omega^2$

e) $(\omega+2)\cdot 2=\omega\cdot 2+2$

f) $2\cdot(\omega+2)=2\cdot\omega+2\cdot 2=\omega+4$

g) $(\omega+2)\cdot\omega=\omega\cdot\omega=\omega^2$

h) $(\omega+2)\cdot(\omega+2)=(\omega+2)\cdot\omega+(\omega+2)\cdot 2=\omega^2+\omega\cdot 2+2$ [by (g) and (e)]

i) $(\omega+2)^2=\omega^2+\omega\cdot 2+2$ [by (h)]

j) $2^{\omega+2}=2^\omega\cdot 2^2=\omega\cdot 4$

k) $(\omega+2)^\omega=\omega^\omega$

l) $(\omega+2)^{\omega+2}=(\omega+2)^\omega\cdot(\omega+2)^2=\omega^\omega\cdot(\omega^2+\omega\cdot 2+2)=\omega^{\omega+2}+\omega^{\omega+1}\cdot 2+\omega^\omega\cdot 2$ [by (k) and (i)]

m) $(\omega^{\omega^\omega})^{\omega^{\omega^\omega}}=\omega^{\omega^\omega\cdot\omega^{\omega^\omega}}=\omega^{\omega^{\omega+\omega^\omega}}=\omega^{\omega^{\omega^\omega}}$

n) $(\omega^{\omega^{\omega^\omega}})^{\omega^{\omega^\omega}}=\omega^{\omega^{\omega^\omega}\cdot\omega^{\omega^\omega}}=\omega^{\omega^{\omega^\omega+\omega^\omega}}=\omega^{\omega^{\omega^\omega\cdot 2}}$

o) $2^{\omega^2}=(2^\omega)^\omega=\omega^\omega$

Remark. It is essential to distinguish between $2+\omega$ (which is ω) and $\omega+2$ (which is not). One cannot do anything with ordinals without having internalized this distinction (made it a part of oneself).

Some people misremembered various rules of arithmetic, or forgot the special conditions under which they apply. I don't try to memorize them, myself; I figure out what they must be, and I play with them, so that I develop a feeling for *why* they should be true.

Problem 2.

a) State the recursive definition of ordinal addition.

b) Prove from this definition that $0 + \alpha = \alpha$ *for all* α.

Solution.

a) $\alpha + 0 = \alpha$

$\alpha + \beta' = (\alpha + \beta)'$

$\alpha + \gamma = \sup\{\alpha + x \colon x < \gamma\}$ if γ is a limit

b) $0 + 0 = 0$.

If $0 + \alpha = \alpha$, then $0 + \alpha' = (0 + \alpha)' = \alpha'$.

If β is a limit, and $0 + \alpha = \alpha$ whenever $\alpha < \beta$, then

$$0 + \beta = \sup_{\alpha<\beta}(0 + \alpha) = \sup_{\alpha<\beta} \alpha = \beta.$$

Problem 3. *Prove that*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

for all ordinals. Use only the recursive *definitions of* $+$ *and* $\cdot$. *You may use the normality of the functions* $x \mapsto \alpha + x$ *and* $x \mapsto \beta \cdot x$ *where* $\beta > 0$, *and you may use the theorem that makes normality useful (as the instructions above suggest).*

Solution. We use induction on γ.

i) $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + 0 = \alpha \cdot \beta + \alpha \cdot 0$.

ii) If $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$, then

$$\begin{aligned} \alpha \cdot (\beta + \gamma') &= \alpha \cdot (\beta + \gamma)' \\ &= \alpha \cdot (\beta + \gamma) + \alpha \\ &= (\alpha \cdot \beta + \alpha \cdot \gamma) + \alpha \\ &= \alpha \cdot \beta + (\alpha \cdot \gamma + \alpha) \\ &= \alpha \cdot \beta + \alpha \cdot \gamma'. \end{aligned}$$

iii) If δ is a limit, and $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ whenever $\gamma < \delta$, then

$$\begin{aligned} \alpha \cdot (\beta + \delta) = \alpha \cdot \sup_{\gamma<\delta}(\beta + \gamma) &= \sup_{\gamma<\delta}(\alpha \cdot (\beta + \gamma)) \\ &= \sup_{\gamma<\delta}(\alpha \cdot \beta + \alpha \cdot \gamma) \\ &= \alpha \cdot \beta + \sup_{\gamma<\delta}(\alpha \cdot \gamma) = \alpha \cdot \beta + \alpha \cdot \delta \end{aligned}$$

Remark. The induction must be on γ; nothing else works. With ordinals, all of the action that we know about from definitions happens on the *right.*

Problem 4. *Assuming $\alpha > 1$, prove that the function $x \mapsto \alpha^x$ on* **ON** *is normal.*

Solution. By definition, if β is a limit, then $\alpha^\beta = \sup\{\alpha^x : x < \beta\}$. Therefore it remains to show

$$\beta < \gamma \Rightarrow \alpha^\beta < \alpha^\gamma.$$

We prove this for all β, by induction on γ.

i) The claim is vacuously true when $\gamma = 0$ [since it is never true that $\beta < 0$].

ii) Suppose the claim is true when $\gamma = \delta$. If $\beta < \delta'$, then $\beta \leqslant \delta$, so

$$\alpha^\beta \leqslant \alpha^\delta < \alpha^\delta \cdot \alpha = \alpha^{\delta'}.$$

iii) Suppose δ is a limit, and the claim holds when $\gamma < \delta$. If now $\beta < \delta$, then $\beta < \beta' < \delta$, so

$$\alpha^\beta < \alpha^{\beta'} \leqslant \sup_{\gamma<\delta} \alpha^\gamma = \alpha^\delta.$$

Scores:

	EA	PC	AF	Mİ	MM	OŞ	NT	ÖT
1	10	13	8	2	7	13	13	9
2	2	5	5	2	0	5	5	4
3	0	5	5	0	0	0	5	0
4	2	0	2	0	0	4	2	2
	14	23	20	4	7	22	25	15