MATH 304, 2009/10, FINAL EXAMINATION *SOLUTIONS*

DAVID PIERCE

Solution 1. **A.** The substitution $x = t - 1$ converts $(*)$ into $(\dagger)$; so x is a solution to $(*)$ if and only if $x + 1$ is a solution to $(\dagger)$).

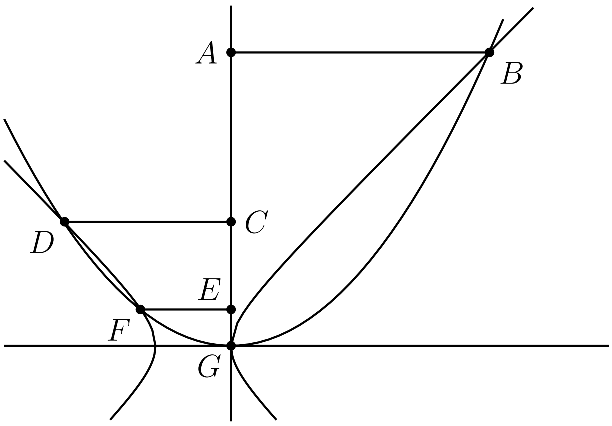
B. From $(\dagger)$ we have

$$\frac{t^2}{9} = \frac{t+1}{t}, \qquad \frac{t}{3} = \frac{y}{t} = \frac{t+1}{y}$$

for some y; that is, we can solve $(\dagger)$ by simultaneously solving $t/3 = y/t$ and $y/t = (t+1)/y$, that is,

$$t^2 = 3y, \qquad y^2 = t(t+1).$$

These equations define a parabola and a hyperbola, respectively, as below. Then AB is a solution to $(\dagger)$.



C. The negative solutions of $(\dagger)$ are CD and EF. (The parabola and hyperbola intersect also at G, but no solution to $(\dagger)$ corresponds to this, since the corresponding value of y is 0.)

D. Let $t = u + v$; then

$$t^3 = 3uvt + u^3 + v^3.$$

Then $(\dagger)$ holds, provided $uv = 3$ and $u^3 + v^3 = 9$. Solving these, we have

$$u^6 + u^3v^3 = 9u^3, \qquad u^6 + 27 = 9u^3, \qquad u^3 = \frac{9}{2} \pm \sqrt{\frac{81}{4} - 27} = \frac{9 \pm 3\sqrt{-3}}{2}.$$

So if u is a cube root of $(9 + 3\sqrt{-3})/2$, then one solution to $(\dagger)$ is $u + 3/u$.

Remark. Cardano could not give a meaning to the solution we found in the last part; today we can, and the three choices of the cube root give the three solutions found geometrically earlier.

Date: June 14, 2010.

Solution 2. **A.** If (‡) holds, then in particular it holds when C is F. Therefore (‡) is equivalent to
$$\frac{CD^2}{AC \times CB} = \frac{EF^2}{AF \times FB} = \frac{a^2}{b^2}, \qquad \frac{x^2}{b^2 - y^2} = \frac{a^2}{b^2}, \qquad b^2x^2 = a^2b^2 - a^2y^2,$$
which is equivalent to (‖).

B. Let $HG = e$. Then (§) becomes
$$\frac{e-b}{e+b} = \frac{b-d}{b+d},$$
which yields $e = b^2/d$.

C. Since $CDL \sim FEG$, and
$$FEG = \frac{1}{2}\Big(\frac{b^2}{d} - d\Big)c,$$
we have
$$CDL = \frac{x^2}{c^2}FEG = \frac{x^2}{2c}\Big(\frac{b^2}{d} - d\Big).$$
We assume angle BAK is right; otherwise, we can just multiply throughout by its sine.) Also AHK and CHM are both similar to FHE, which is $cd/2$; so
$$AKH - CHM = \frac{cd}{2}\Big(\frac{b^2}{d^2} - \frac{y^2}{d^2}\Big).$$
So (∗∗) holds if and only if
$$\frac{x^2}{c}\Big(\frac{b^2}{d} - d\Big) = c\frac{b^2 - y^2}{d},$$
$$x^2(b^2 - d^2) = c^2(b^2 - y^2),$$
$$b^2x^2 = a^2(b^2 - y^2),$$
which is equivalent to (‖).

D. In (∗∗), let C be F; then the equation becomes
$$FEG = AHK - FHE,$$
so $AHK = FEG + FHE = GHE$.

E. By part **C**, it is enough to show
$$PDM = EHG - PHL.$$
We have
$$\begin{aligned} PDM &= CDL + CHM - PHL \\ &= AHK - PHL && \text{[by (∗∗)]} \\ &= EHG - PHL && \text{[by } \mathbf{D}\text{].} \end{aligned}$$

Mathematics Dept., Middle East Technical University, Ankara 06531, Turkey
E-mail address: dpierce@metu.edu.tr
URL: http://metu.edu.tr/~dpierce/