MATH 304 EXAMINATION SOLUTIONS

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You may use modern notation in your work; but Problems 2 and 3 should involve diagrams.

Problem 1. *A straight line is cut into equal and unequal segments. What is the relationship between the square on the half and the rectangle contained by the unequal segments?*

Solution. The square exceeds the rectangle by the square on the straight line between the points of section.

Remark. This problem is based on Proposition II.5 of Euclid's *Elements.* The language follows the style of Heath's translation of Euclid (on the course webpage).

Problem 2. *A square is equal to three roots and twenty-eight dirhams. What is the root? Give a geometrical justification of your answer (as Muḥammad ibn Mūsā al-Khwārizmī or Thābit ibn Qurra did).*

Solution. In Figure 1, the root is AB; $AC = 3$; and D bisects AC. Then

$$DB^2 = 28 + DC^2 = 28 + \left(\frac{3}{2}\right)^2 = \frac{121}{4},$$

$$DB = \frac{11}{2},$$

$$AB = AD + DB = \frac{3}{2} + \frac{11}{2} = 7;$$

so the root is 7.

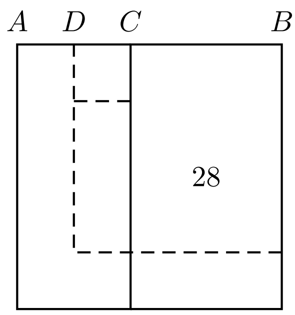

FIGURE 1

Remark. Euclid's Proposition II.6 is behind this problem.

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Problem 3. *Suppose a cube and nine sides are equal to ten. Find the side by taking the intersection of two conic sections (as Omar Khayyām did). It is preferable if one of those sections is a circle.*

Solution. [*Analysis:*]

$$x^3 + 9x = 10,$$

$$x^3 = 10 - 9x,$$

$$\frac{x^2}{9} = \frac{10/9 - x}{x}, \tag{$*$}$$

$$\frac{x}{3} = \frac{y}{x} = \frac{10/9 - x}{y}, \tag{$\dagger$}$$

$$x^2 = 3y \quad \& \quad y^2 = x\left(\frac{10}{9} - x\right).$$

[*Synthesis:*] As in Figure 2, let ABC be a semicircle with diameter $10/9$, and let AD, perpendicular to AB, be the axis of a parabola with parameter 3. The semicircle and parabola intersect at a point C (as well as at A). Let CE be dropped perpendicular to AB; and CD, to AD. Then $AE = CD$; either of these is the desired "side". Indeed,

$$CD^2 = 3AD,$$

$$CD : 3 :: AD : CD :: EC : AE :: EB : EC,$$

$$AE^2 : 9 :: CD^2 : 9 :: EB : AE :: \left(\frac{10}{9} - AE\right) : AE,$$

$$AE^3 = 10 - 9AE,$$

$$AE^3 + 9AE = 10.$$

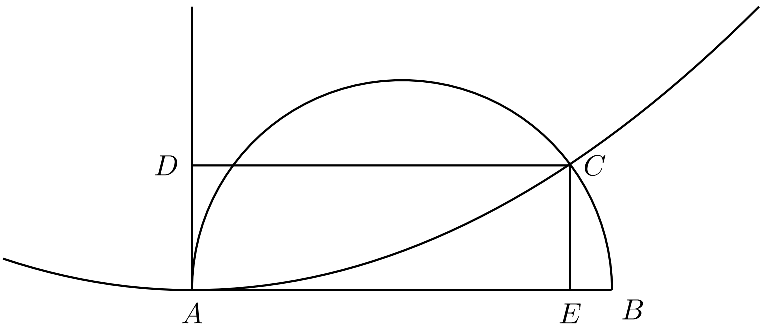

FIGURE 2

Remark. (i). In the solution, *analysis* and *synthesis* are used in the sense attributed to Theon (presumably Theon of Smyrna, that is, İzmir) by Viète at the beginning of Chapter 1 of the *Introduction to the Analytic Art.* In his solutions of cubic equations, Omar Khayyām gives only the synthesis; we can only speculate whether he had some sort of analysis like ours.

(ii). In our analysis, equations (∗) and (†) could have been

$$x^2 = \frac{10-9x}{x},$$
$$x = \frac{y}{x} = \frac{10-9x}{y},$$

yielding the parabola given by $y = x^2$ and the ellipse given by $y^2 = x(10-9x)$. This is why the problem says, "It is preferable if one of those sections is a circle."

(iii). I think it is better to understand the circle through the equation $y^2 = x(10/9-x)$ than to convert this equation to the more usual modern form,

$$y^2 + \left(x - \frac{5}{9}\right)^2 = \left(\frac{5}{9}\right)^2.$$

Problem 4. *Again, a cube and nine sides are equal to ten.*

- *(a). Find the side numerically, as the difference of the cube roots of a* binomium *and an* apotome, *by Cardano's method (really Tartaglia's method); your steps should be clearly justifiable.*
- *(b). The side is in fact a whole number; which one?*

Solution. (a). We have to solve $x^3 + 9x = 10$. We let $x = u - v$, so

$$x^3 = u^3 - v^3 - 3uv(u-v) = u^3 - v^3 - 3uvx.$$

So we let

$$u^3 - v^3 = 10, \qquad uv = 3,$$

which we can solve:

$$u^6 - u^3v^3 = 10u^3,$$
$$u^6 - 27 = 10u^3,$$
$$u^3 = \sqrt{5^2+27} + 5 = 2\sqrt{13} + 5,$$
$$v^3 = \frac{3^3}{2\sqrt{13}+5} = 2\sqrt{13} - 5.$$

Therefore

$$x = \sqrt[3]{2\sqrt{13}+5} - \sqrt[3]{2\sqrt{13}-5}.$$

(b). $x = 1$.

Remark. (i). Cardano does give a formula for finding x, without clear explanation. However, this problem said "steps should be clearly justifiable"; so for full credit, the answer should be *derived,* as above, not just obtained from a memorized formula. Some people who tried to memorize, remembered wrongly.

(ii). Of course, the solution above did rely on the (memorized) quadratic formula. Memory does have its uses.

(iii). Note here that u^3 could have been $-2\sqrt{13}+5$; but x in the end would have been the same. Two other values of x can be obtained by considering *complex* cube roots; but Cardano does not know about these.

Problem 5. *A square-square, twelve squares, and thirty-six are equal to seventy-two sides. In finding the side by Cardano's method (really Ferrari's method), you first solve a cubic equation.*

(a). Obtain that cubic equation.
(b). Convert that cubic equation to an equation of the form "cube equal to roots and number".
(c). The cubic equation in (a) *should have* 6 *as a root. Use this to find the side in the original fourth-degree equation.*

Solution. (a).
$$\begin{aligned} x^4+12x^2+36&=72x,\\ (x^2+6)^2&=72x,\\ (x^2+6+t)^2&=2tx^2+72x+t^2+12t,\\ 2t(t^2+12t)&=36^2=2^43^4,\\ t^3+12t^2&=2^33^4=648. \end{aligned}$$

(b). Let $t=s-4$; then
$$s^3-48s+12\cdot 16-64=2^33^4,$$
$$s^3-48s=2^33^4+2^6-2^63=2^3(3^4-2^4)=8\cdot 65=520.$$

(c).
$$\begin{aligned} (x^2+12)^2&=12x^2+72x+108,\\ &=12(x^2+6x+9)\\ &=12(x+3)^2,\\ x^2+12&=2\sqrt{3}(x+3),\\ x^2&=2\sqrt{3}-6(2-\sqrt{3}),\\ x&=\sqrt{3}+\sqrt{3-6(2-\sqrt{3})}=\sqrt{3}+\sqrt{6\sqrt{3}-9}. \end{aligned}$$

Remark. If we believe in negative numbers, then from $(x^2+12)^2=12(x+3)^2$ we should obtain $x^2+12=\pm 2\sqrt{3}(x+3)$; but the negative sign here leads to a negative value of x. The problem asks for the "side", which is implicitly positive.

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