# How many rational points can a high genus curve over a finite field have?

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# Affine plane curves

$k$ a perfect field (e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q \dots$)
$\bar{k}$ a fixed algebraic closure of $k$
Let $f(X, Y) \in k[X, Y]$.
The affine plane curve defined by $f(X, Y)$:

$$\mathcal{C}_f := \{(x, y) \in \bar{k} \times \bar{k} | f(x, y) = 0\}$$

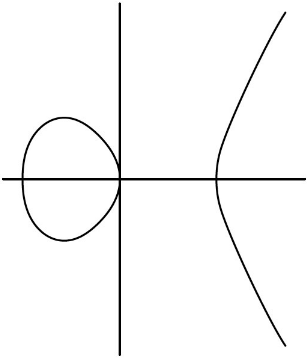
$\mathcal{C}_f$ is defined over $k$.
The set of $k$-rational points of $\mathcal{C}_f$:

$$\mathcal{C}_f(k) := \{(x, y) \in k \times k | f(x, y) = 0\}$$

# An example

$k = \mathbb{R}$

$$f(X, Y) = Y^2 - X \cdot (X - 1) \cdot (X + 1).$$



$\mathcal{C}_f(\mathbb{R})$

# Curves in $n$-space

Can generalize this to curves in higher dimensional space: $\mathcal{C} \subset \bar{k}^n$
$f_1, f_2, \dots f_{n-1} \in k[X_1, X_2, \dots, X_n]$.
Affine curve:

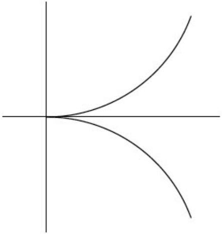
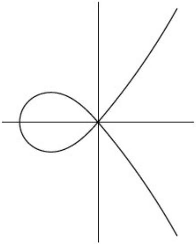
$$\mathcal{C} := \{(a_1, \dots, a_n) \in \bar{k}^n | f_i(a_1, \dots, a_n) = 0 \quad \text{for} \quad i = 1, 2, \dots n-1\}$$

The set of $k$-rational points of $\mathcal{C}$:

$$\mathcal{C}(k) := \{(a_1, \dots, a_n) \in k^n | f_i(a_1, \dots, a_n) = 0 \quad \text{for} \quad i = 1, 2, \dots n-1\}$$

From now on we assume that $\mathcal{C}$ is a

- absolutely irreducible
- smooth

- projective

curve defined over $k$.

# The genus

Invariant

$$g(\mathcal{C}) : \text{ a nonnegative integer}$$

$\mathcal{C}$ is a line/conic $\longrightarrow$ genus 0

$\mathcal{C}$ is an elliptic curve $\longrightarrow$ genus 1

## Curves over Finite Fields

From now on $k = \mathbb{F}_q$

$\mathcal{C}/\mathbb{F}_q \to \mathcal{C} \subset \bar{\mathbb{F}_q}^n$ for some $n \in \mathbb{N}$

$$\mathcal{C}(\mathbb{F}_q) \subset \mathbb{F}_q^n$$

So

$$\#\mathcal{C}(\mathbb{F}_q) \text{ is finite}$$

$$\#\mathcal{C}(\mathbb{F}_q) = ?$$

# The Hasse–Weil bound

$\mathcal{C} \longrightarrow \zeta_{\mathcal{C}}$ Zeta function of $\mathcal{C}$

**Theorem (Hasse–Weil)**

*The Riemann hypothesis holds for $\zeta_{\mathcal{C}}$.*

**Corollary (Hasse–Weil bound)**

*Let $\mathcal{C}/\mathbb{F}_q$ be a curve of genus $g(\mathcal{C})$. Then*

$$\#\mathcal{C}(\mathbb{F}_q) \leq q + 1 + 2\sqrt{q} \cdot g(\mathcal{C}).$$

# How good is the Hasse–Weil bound?

Various improvements, but:
If the genus $g(\mathcal{C})$ is small (with respect to $q$) $\longrightarrow$ Hasse–Weil bound is good.
It can be attained, *maximal curves*, for example over $\mathbb{F}_{q^2}$

$$y^q + y = x^{q+1}.$$

Ihara, Manin: The Hasse–Weil bound can be improved if $g(\mathcal{C})$ is large (with respect to $q$).

# Ihara's constant

Ihara:

$$A(q) = \limsup_{g(\mathcal{C})\to\infty} \frac{\#\mathcal{C}(\mathbb{F}_q)}{g(\mathcal{C})}$$

$\mathcal{C}$ runs over all absolutely irreducible, smooth, projective curves over $\mathbb{F}_q$.

| | |
|---|---|
| Hasse–Weil bound | $\Longrightarrow A(q) \leq 2\sqrt{q}$ |
| Ihara | $\Longrightarrow A(q) \leq \sqrt{2q}$ |
| Drinfeld–Vladut | $\Longrightarrow A(q) \leq \sqrt{q} - 1$ |

# Lower bounds for $A(q)$

Serre (using class field towers):

$$A(q) > 0$$

Ihara (modular curves):
If $q = \ell^2$ then

$$A(q) \geq \sqrt{q} - 1 = \ell - 1$$

In fact $A(\ell^2) = \ell - 1$.

Zink (Shimura surfaces):
If $q = p^3$, $p$ a prime number, then

$$A(p^3) \geq \frac{2(p^2 - 1)}{p + 2}$$

(generalized by Bezerra–Garcia–Stichtenoth to all cubic finite fields)

# How to obtain lower bounds for $A(q)$?

Find sequences $\mathcal{C}_i/\mathbb{F}_q$ such that $g(\mathcal{C}_i) \to \infty$ and

$$\lim_{i\to\infty} \frac{\#\mathcal{C}_i(F_q)}{g(\mathcal{C}_i)} \text{ is large.}$$

Many ways to construct good sequences:

- Modular curves (Elliptic, Shimura, Drinfeld) (over $\mathbb{F}_{q^2}$)
- Class field towers (over prime fields)
- Explicit equations (recursively defined)

# Recursively defined towers

$f_1, f_2, \dots f_{n-1} \in \mathbb{F}_q[X_1, X_2, \dots, X_n]$

$\mathcal{C} := \{(a_1, \dots, a_n) \in \bar{\mathbb{F}_q}^n | f_i(a_1, \dots, a_n) = 0 \text{ for } i = 1, 2, \dots n-1\}$

Recursively defined tower:
Fix $F(U, V) \in \mathbb{F}_q[U, V]$.
Define

$$f_1 = F(X_1, X_2)$$

$$f_2 = F(X_2, X_3)$$

$$\dots$$

$$f_{n-1} = F(X_{n-1}, X_n)$$

$$\mathcal{C}_n := \{(a_1, \dots, a_n) \in \bar{\mathbb{F}_q}^n | f_1 = f_2 = \dots = f_{n-1} = 0\}$$

$\mathcal{F} = (\mathcal{C}_n)_{n \geq 1}$ tower recursively defined by $F$.

Recursively defined by $f(U, V) \in \mathbb{F}_q[U, V]$

$$
\begin{array}{c}
C_4 = \{(a_1, a_2, a_3, a_4) | F(a_1, a_2) = F(a_2, a_3) = F(a_3, a_4) = 0\} \subseteq \bar{\mathbb{F}}_q^{\,4} \\
\big\downarrow \\
C_3 = \{(a_1, a_2, a_3) | F(a_1, a_2) = 0, F(a_2, a_3) = 0\} \subseteq \bar{\mathbb{F}}_q^{\,3} \\
\big\downarrow \\
C_2 = \{(a_1, a_2) | F(a_1, a_2) = 0\} \subseteq \bar{\mathbb{F}}_q^{\,2}
\end{array}
$$

# Limit of a tower

Limit of the tower $\mathcal{F} = (\mathcal{C}_n)_{n\geq 1}$ over $\mathbb{F}_q$

$$\lambda(\mathcal{F}) = \lim_{n\to\infty} \frac{\#\mathcal{C}_n(\mathbb{F}_q)}{g(\mathcal{C}_n)} \leq A(q) \leq \sqrt{q} - 1$$

exists

$\lambda(\mathcal{F}) = 0 \longrightarrow$ asymptotically bad

$\lambda(\mathcal{F}) > 0 \longrightarrow$ asymptotically good

# Example

Garcia–Stichtenoth, 1996, Norm-Trace tower $\mathcal{F}_1$
$q = \ell^2$

$$V^\ell + V = \frac{U^{\ell+1}}{U^\ell + U}$$

$$\lambda(\mathcal{F}_1) = \sqrt{q} - 1$$

Attains the Drinfeld–Vladut bound.
Genus computation is difficult (wild ramification)
Why many rational points?

$$q = \ell^2 \qquad V^\ell + V = \frac{U^{\ell+1}}{U^\ell + U}$$

$$X_n^\ell + X_n = \frac{X_{n-1}^{\ell+1}}{X_{n-1}^\ell + X_{n-1}}, \ldots, X_3^\ell + X_3 = \frac{X_2^{\ell+1}}{X_2^\ell + X_2},\ X_2^\ell + X_2 = \frac{X_1^{\ell+1}}{X_1^\ell + X_1}$$

$$X_1 = a_1 \in \mathbb{F}_q \text{ s.t. } Tr_{\mathbb{F}_q/\mathbb{F}_\ell}(a_1) \neq 0$$

($\ell^2 - \ell$ choices)

$$X_2 = a_2 \text{ with } a_2^\ell + a_2 = \frac{a_1^{\ell+1}}{a_1^\ell + a_1} \in \mathbb{F}_\ell \setminus \{0\}$$

$\ell$ choices with $a_2 \in \mathbb{F}_q$, $Tr_{\mathbb{F}_q/\mathbb{F}_\ell}(a_2) \neq 0$)

$$X_3 = a_3 \text{ with } a_3^\ell + a_3 = \frac{a_2^{\ell+1}}{a_2^\ell + a_2} \in \mathbb{F}_\ell \setminus \{0\}$$

$\ell$ choices with $a_3 \in \mathbb{F}_q$, $Tr_{\mathbb{F}_q/\mathbb{F}_\ell}(a_3) \neq 0$)

$\cdots \;\; \cdots$ so $\#\mathcal{C}_n(\mathbb{F}_q) \geq (\ell^2 - \ell)\ell^{n-1}$

# Towers over cubic finite fields

- van der Geer–van der Vlugt, $\quad q = 2^3 = 8, \mathcal{F}_2/\mathbb{F}_q$

$$V^2 + V = U + 1 + 1/U$$

  Attains Zink's bound for $p = 2$.

- Bezerra–Garcia–Stichtenoth, $\quad q = \ell^3, \mathcal{F}_3/\mathbb{F}_q$

$$\frac{1 - V}{V^\ell} = \frac{U^\ell + U + 1}{U} \quad \lambda(\mathcal{F}_3) \geq \frac{2(\ell^2 - 1)}{\ell + 2}.$$

  Generalizes Zink's bound.

- B.–Garcia–Stichtenoth, $\quad q = \ell^3, \mathcal{F}_4/\mathbb{F}_q$

$$(V^\ell - V)^{\ell-1} + 1 = \frac{-U^{\ell(\ell-1)}}{(U^{\ell-1} - 1)^{\ell-1}} \quad \lambda(\mathcal{F}_4) \geq \frac{2(\ell^2 - 1)}{\ell + 2}.$$

# A new family of towers over all non-prime fields

B.–Beelen–Garcia–Stichtenoth
$\mathcal{F}_5$ over $\mathbb{F}_{\ell^n}, n \geq 2$:

Notation: $Tr_n(t) = t + t^\ell + \cdots + t^{\ell^{n-1}}, \; N_n(t) = t^{1+\ell+\ell^2+\ldots+\ell^{n-1}}$

$$\frac{N_n(V)+1}{V^{\ell^{n-1}}} = \frac{N_n(U)+1}{U}.$$

Splitting: $N_n(\alpha) = -1$

$$\lambda(\mathcal{F}_5) \geq \frac{2}{\frac{1}{\ell-1} + \frac{1}{\ell^{n-1}-1}}$$

- $n = 2$: $\ell - 1 \longrightarrow$ Drinfeld-Vladut bound
- $n = 3$: $\frac{2(\ell^2-1)}{\ell+2} \longrightarrow$ Zink's bound

$\mathcal{F}_6/\mathbb{F}_q,\ q = \ell^n,\ n = 2k+1 \geq 3$

$$\frac{Tr_k(V) - 1}{(Tr_{k+1}(V) - 1)^{\ell^k}} = \frac{(Tr_k(U) - 1)^{\ell^{k+1}}}{(Tr_{k+1}(U) - 1)}$$

$$\frac{V^{\ell^n} - V}{V^{\ell^k}} = -\frac{(1/U)^{\ell^n} - (1/U)}{U^{\ell^{k+1}}}$$

$\mathcal{F}_6/\mathbb{F}_q, \; q = \ell^n, \; n = 2k+1$

$$\lambda(\mathcal{F}_6) \geq \frac{2}{\frac{1}{\ell^k - 1} + \frac{1}{\ell^{k+1}-1}} \geq \frac{2(\ell^{k+1}-1)}{\ell + 1 + \epsilon}$$

with

$$\epsilon = \frac{\ell - 1}{\ell^k - 1}.$$

Note:

$$\ell^{k+\frac{1}{2}} - 1 \geq A(\ell^{2k+1}) \geq \frac{2}{\frac{1}{\ell^k-1} + \frac{1}{\ell^{k+1}-1}}.$$

$2^{15} \quad (2^3)^5 \quad (2^5)^3$
$q = 2^k$, $k$ large,

$$\frac{\lambda(\mathcal{F}_5)}{\sqrt{q} - 1} \approx 94\%$$

# Elliptic Curves

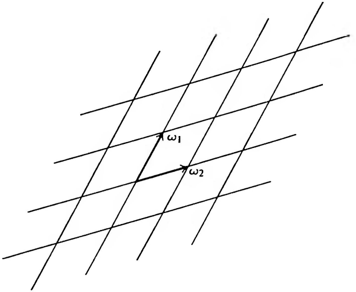
$E/k,\ \ char(k) \neq 2, 3$

$$E:\ \ Y^2 = X^3 + A \cdot X + B,$$

where $4A^3 + 27B^2 \neq 0$.

# Elliptic Curves over $\mathbb{C}$

$k = \mathbb{C}$

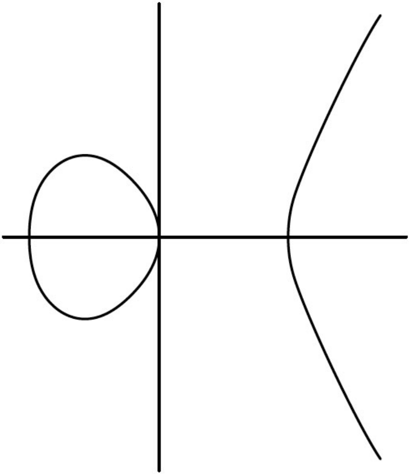



$$\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}.$$

$\mathbb{C}/\Lambda$
topologically a torus
inherits a complex structure from $\mathbb{C}$.
Complex manifold $\rightarrow E(\mathbb{C})$

# The group law

Points in $E$ inherit a group structure from $\mathbb{C}$:



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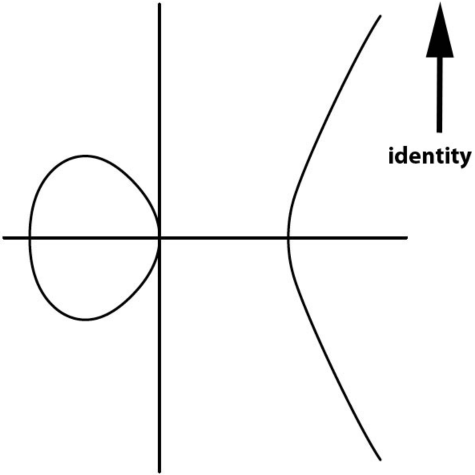

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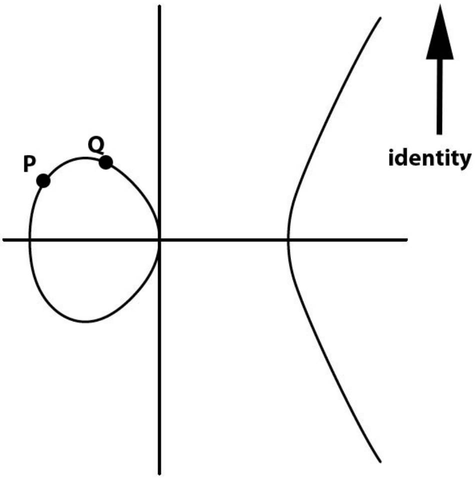

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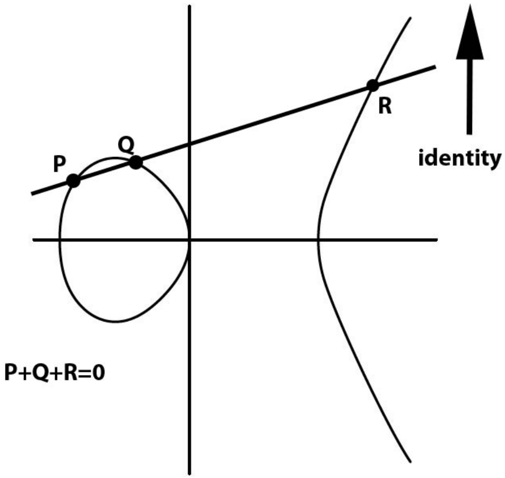

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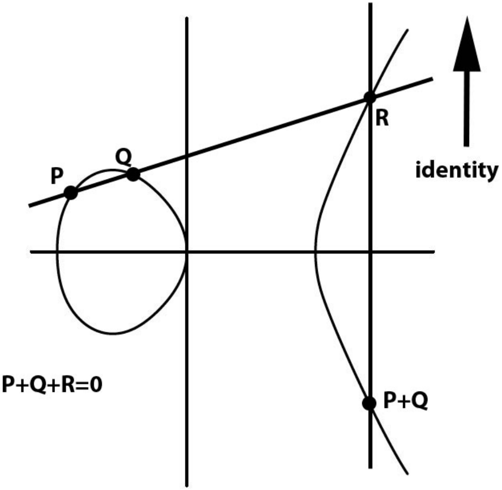


# Isogenies

A morphism $\varphi : \; E_1 \to E_2$, which is a group homomorphism is called an *isogeny*.

*Example:* $E$ elliptic curve, $N \in \mathbb{N}$

$$\begin{array}{rccc} [N]: & E & \to & E \\ & P & \mapsto & \underbrace{P + P + \ldots P}_{N \text{ times}} \end{array}$$

$$\# ker(\varphi) \text{ is finite.}$$

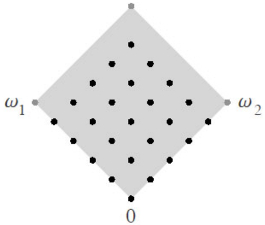
$\# ker(\varphi) = N \to \varphi$ is an $N$-isogeny $\to ker(\varphi) \subset ker([N])$.

# Torsion

$ker([N]) = \{P \in E | N \cdot P = 0\} =: E[N] \;\; \rightarrow$ $N$-torsion points

if $char(k) \nmid N \quad E[N] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

if $char(k) = p \quad E[p] \cong$ $\{0\} \rightarrow$ supersingular or $\mathbb{Z}/p\mathbb{Z} \rightarrow$ ordinary



# Isomorphism classes of elliptic curves

$\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic

$$\iff$$

$\Lambda_1$ and $\Lambda_2$ are homothetic, i.e. $\Lambda_1 = \alpha\Lambda_2, \alpha \in \mathbb{C}^\times$.

Let

$$\mathbb{H} = \{\tau \in \mathbb{C} | Im(\tau) > 0\}.$$

Every lattice is homothetic to a lattice of the form

$$\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$$

with $\tau \in \mathbb{H}$.
When are $\Lambda_\tau$ and $\Lambda_{\tau'}$ the same lattice?

When are $\Lambda_\tau$ and $\Lambda_{\tau'}$ the same lattice?

$\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

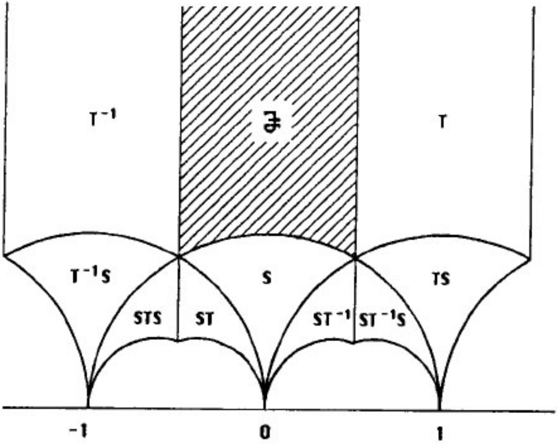
$\Lambda_\tau$ and $\Lambda_{\tau'}$ are the same lattice

$$\Longleftrightarrow$$

$\tau$ and $\tau'$ are in the same orbit under the action of $\mathrm{SL}_2(\mathbb{Z})$.

# Isomorphism classes of elliptic curves

Elliptic curves / isomorphism $\longleftrightarrow$ lattices in $\mathbb{C}$ / homothety $\longleftrightarrow \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \quad \to X(1)$



# The $j$-Function

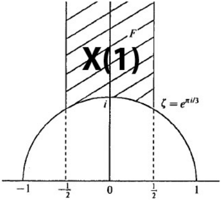
There exists a holomorphic function

$$j : \mathbb{H} \to \mathbb{C},$$

which is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

$$j : \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$$

is a bijection!




$\longrightarrow$ $j$-line

$[E] \longrightarrow j$-invariant

Fact: $E$ supersingular $\longrightarrow j(E) \in \mathbb{F}_{p^2}$,
where $p$ is the characteristic.

$\therefore$ $j$-line parametrizes isomorphism classes of Elliptic curves
$\rightarrow$ has designated $\mathbb{F}_{p^2}$-rational points.

# Enhanced Elliptic Curves

Elliptic curves with some additional structure

$$(E, C)$$

$E$: Elliptic Curve
$C$: cyclic subgroup of order $N$ / $N$-isogeny

$$(E, C) \sim (E', C') \quad \text{isomorphism takes } C \to C'.$$

$X_0(N)$ modular curve parametrizing $(E, C)$.

$$\begin{tikzcd}
X_0(N) \arrow[d, "\text{forget}"] & & A \subset X_0(N)(\mathbb{F}_{p^2}) \arrow[d] \\
X(1) & \supset & \text{supersingular} \subset X(1)(\mathbb{F}_{p^2})
\end{tikzcd}$$

$(N_i)_{i \geq 0}$ with $N_i \to \infty$, $p \nmid N_i$.

$$C_{N_i} = (X_0(N_i) \pmod{p})$$

- $\#C_{N_i}(\mathbb{F}_{p^2})$ is large (supersingular points)
- $g(C_{N_i})$ can be estimated

$$\frac{\#C_{N_i}(\mathbb{F}_{p^2})}{g(C_{N_i})} \to \sqrt{p^2} - 1 = p - 1 \quad \text{(Drinfeld-Vladut bound)}$$

Elkies: $X_0(\ell^n)$ recursive.

# Drinfeld Modular Varieties

$$
\begin{array}{ccc}
 & & C_\infty \\
 & & | \\
\mathbb{C} & & K \\
| & & | \\
\mathbb{R} & & R \\
| & & | \\
\mathbb{Q} & & \mathbb{F}_q(T) \\
| & & | \\
\mathbb{Z} & & \mathbb{F}_q[T]
\end{array}
$$

$\mathbb{Z}$-lattices inside $\mathbb{C}$ $\quad\longrightarrow$ rank 1 or 2

$\mathbb{F}_q[T]$-lattices inside $C_\infty$ $\quad\longrightarrow$ arbitrary high rank possible

## Drinfeld Modular Curves

$A = \mathbb{F}_\ell[T]$, $P$ a prime of $A$,

$$\mathbb{F}_P = A/ < P >= \mathbb{F}_{\ell^d}$$

where $d = \deg P$.
$\mathbb{F}_P^{(2)}$: The unique quadratic extension of $\mathbb{F}_P$.
For $N \in \mathbb{F}_\ell[T]$ we have

$$X_0(N)$$

an algebraic curve defined over $\mathbb{F}_\ell(T)$, Drinfeld modular curve, parametrizing rank 2 Drinfeld modules together with an $N$-isogeny.
$X_0(N)$ has good reduction at all primes $P \nmid N$.

$$X_0(N)/\mathbb{F}_P$$

# Many points on Drinfeld modular curves

$X_0(N)/\mathbb{F}_P$ has many rational points over $\mathbb{F}_P^{(2)} = \mathbb{F}_{\ell^{2d}}$, where $d = \deg P$. Asymptotically:

## Theorem (Gekeler)

*$P \in \mathbb{F}_\ell[T]$ prime of degree $d$*
*$(N_k)_{k\geq 0}$: sequence of polynomials in $\mathbb{F}_\ell[T]$ with*

- $P \nmid N_k$
- $\deg N_k \to \infty$

*Then the sequence of curves*

$$X_0(N_k)/\mathbb{F}_P$$

*attains the Drinfeld–Vladut bound over $\mathbb{F}_P^{(2)} = \mathbb{F}_{\ell^{2d}}$.*

**Elkies:** $X_0(Q^n)$ recursive.
Norm trace tower is related to (degree $\ell - 1$ cover of)

$$X_0(T^n)/\mathbb{F}_{T-1}$$

# Many points over non-quadratic fields

Many points come from the supersingular points
$\longrightarrow$ defined over $\mathbb{F}_P^{(2)}$.

In general:

## Theorem (Gekeler)

*Any supersingular Drinfeld module $\phi$ of rank $r$ and characteristic $P$ is isomorphic to one defined over $L$, where $L$ is an extension of $F_P$ of degree $r$.*

Idea: Look at space parametrizing rank $r$ Drinfeld modules
Problem: The corresponding space is higher dimensional ($(r-1)$-dimensional), not a curve!
Idea': Look at curves on those spaces, passing through the many $\mathbb{F}_{\ell^r}$-rational points

(B.–Beelen–Garcia–Stichtenoth)

$$\frac{Tr_k(V)-1}{(Tr_{k+1}(V)-1)^{\ell^k}} = \frac{(Tr_k(U)-1)^{\ell^{k+1}}}{(Tr_{k+1}(U)-1)}$$

$\mathcal{F}/\mathbb{F}_q,\ q=\ell^n,\ n=2k+1$

$$A(q) \geq \lambda(\mathcal{F}) \geq \frac{2}{\frac{1}{\ell^k-1}+\frac{1}{\ell^{k+1}-1}} \geq \frac{2(\ell^{k+1}-1)}{\ell+1+\epsilon}$$

with

$$\epsilon = \frac{\ell-1}{\ell^k-1}.$$

joint work (in progress) with Beelen, Garcia, Stichtenoth

Let $\phi$ be a rank $n$ Drinfeld Module of characteristic $T-1$.

$$\phi_T = \tau^n + g_1\tau^{n-1} + g_2\tau^{n-2} + \cdots + g_{n-1}\tau + 1$$

Let $\lambda : \phi \to \psi$ be an isogeny of the form

$$\tau - u$$

whose kernel is annihilated by $T$.

$\exists \mu = \tau^{n-1} + a_2\tau^{n-2} + \cdots + a_{n-1}\tau + a_n$, s.t.

$$\mu \cdot \lambda = \phi_T$$

Then

$$N_n(u) + g_1 \cdot N_{n-1}(u) + g_2 \cdot N_{n-2}(u) + \cdots + g_{n-1} \cdot N_1(u) + 1 = 0$$

Notation: $N_k(x) = x^{1+\ell+\cdots+\ell^{k-2}+\ell^{k-1}}$

# Equations for the isogenous Drinfeld module

$\lambda : \phi \to \psi$

$$\psi_T = \tau^n + h_1 \cdot \tau^{n-1} + \cdots + h_{n-1} \cdot \tau + 1$$

Isogeny: $\lambda \cdot \phi = \psi \cdot \lambda$

$$\begin{aligned} h_{n-1} u^{\ell} &= g_{n-1} u \\ h_{n-2} u^{\ell^2} - h_{n-1} &= g_{n-2} u - g_{n-1}^{\ell} \\ &\cdots\cdots \\ h_1 u^{\ell^{n-1}} - h_2 &= g_1 u - g_2^{\ell} \\ u^{\ell^n} - h_1 &= u - g_1^{\ell} \end{aligned}$$

$$1 + N_n(u)\Big[1 + \frac{h_1}{N_1(u)} + \frac{h_2^{\ell}}{N_2(u)} + \cdots + \frac{h_{n-1}^{\ell^{n-2}}}{N_{n-1}(u)}\Big] = 0.$$

$g_1 = g_2 = \cdots = g_{n-1} = 0 \rightarrow$ supersingular (will split).

Find curve passing through this point and invariant under $g_i \rightarrow h_i$.

Consider $g_2 = \cdots = g_{n-1} = 0 \Rightarrow h_2 = \cdots = h_{n-1} = 0$

$$-g_1 = \frac{N_n(1/u)+1}{(1/u)^{\ell^{n-1}}},$$

$$-h_1 = \frac{N_n(1/u)+1}{1/u}$$

Letting $v_0 = 1/u$

$$\begin{tikzcd}
 & \mathbb{F}_q(v_0) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{F}_q(g_1) & & \mathbb{F}_q(h_1)
\end{tikzcd}$$

$$\frac{N_n(V)+1}{V^{\ell^{n-1}}} = \frac{N_n(U)+1}{U}.$$